Nonhelical inverse transfer of a decaying turbulent magnetic field

Cosmological magnetic fields
Turbulent decay
Nonuniversality of MHD
Weak and strong turbulence
Helical, nonhelical, hydro

Axel Brandenburg (Nordita ←→ CU Boulder)

with T. Kahniashvili and A. Tevzadze

PRL 114, 075001 (2015)

PHYSICAL REVIEW D VOLUME 54, NUMBER 2 15 JULY 1996

Motivation

Early universe

Large-scale magnetic fields from hydromagnetic turbulence in the very early universe

Axel Brandenburg*
Nordita, Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark

Kari Enqvist†
Department of Physics, P.O. Box 9, FIN-00014 University of Helsinki, Finland

Poul Olesen‡
The Niels Bohr Institute, Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark
(Received 1 February 1996)

Energy momentum tensor

$$T^{\mu\nu}=(p+\rho)U^{\mu}U^{\nu}+pg^{\mu\nu}$$
$$+\frac{1}{4\pi}\left(F^{\mu\sigma}F^{\nu}{}_{\sigma}-\frac{1}{4}g^{\mu\nu}F_{\lambda\sigma}F^{\lambda\sigma}\right),$$

Conformal time, rescaled equations

$$\tilde{t}=\int dt/R. \qquad \mathbf{S}=(p+\rho)\gamma^2\mathbf{v}.$$

$$\tilde{\mathbf{S}}=R^4\mathbf{S}, \quad \tilde{p}=R^4p, \quad \tilde{\rho}=R^4\rho, \quad \tilde{\mathbf{B}}=R^2\mathbf{B},$$
$$\tilde{\mathbf{J}}=R^3\mathbf{J}, \quad \text{and} \quad \tilde{\mathbf{E}}=R^2\mathbf{E}.$$

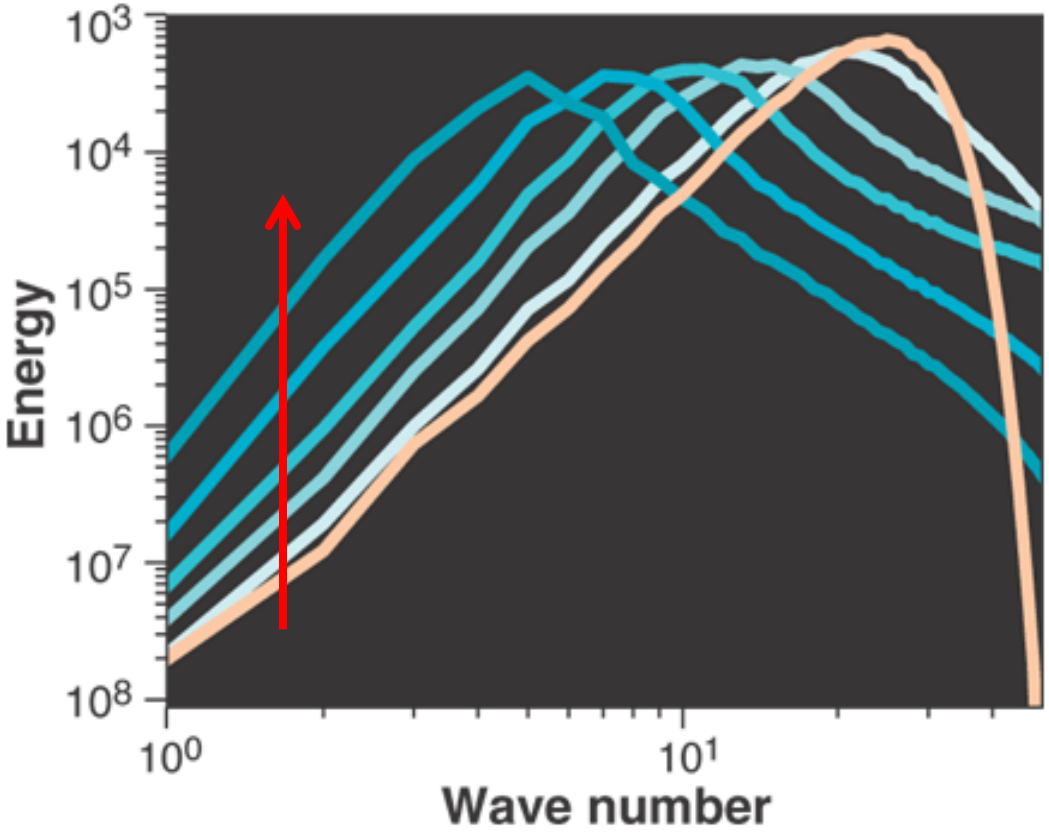
$$\frac{\partial\tilde{\mathbf{S}}}{\partial\tilde{t}}=-(\boldsymbol{\nabla}\cdot\mathbf{v})\tilde{\mathbf{S}}-(\mathbf{v}\cdot\boldsymbol{\nabla})\tilde{\mathbf{S}}-\boldsymbol{\nabla}\tilde{p}+\tilde{\mathbf{J}}\times\tilde{\mathbf{B}}.$$

$$\frac{\partial\tilde{\mathbf{B}}}{\partial\tilde{t}}=-\boldsymbol{\nabla}\times\tilde{\mathbf{E}}, \quad \boldsymbol{\nabla}\cdot\tilde{\mathbf{B}}=0,$$

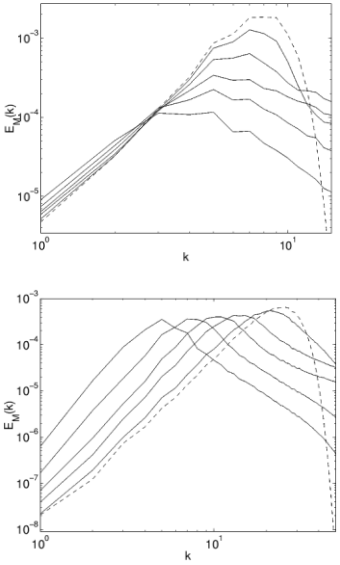
the MHD equations in an expanding universe with zero curvature are the same as the relativistic MHD equations in a nonexpanding universe, provided the dynamical quantities are replaced by the scaled ''tilde'' variables, and provided conformal time $\tilde{t}$ is used. The effect of this is, as usual, that

Helical vs nonhelical 3-D decay

Initial slope
$E \sim k^4$



helical vs
nonhelical



Christensson et al.
(2001, PRE 64, 056405)

Helical decay law:
Biskamp & Müller (1999)

$$H = EL = \text{const}$$

$$\varepsilon = U^3 / L = E^{3/2} / L$$

$$\varepsilon = -\mathrm{d}E / \mathrm{d}t$$

$$\varepsilon = -\mathrm{d}E / \mathrm{d}t = E^{3/2} / L = E^{5/2} / H$$

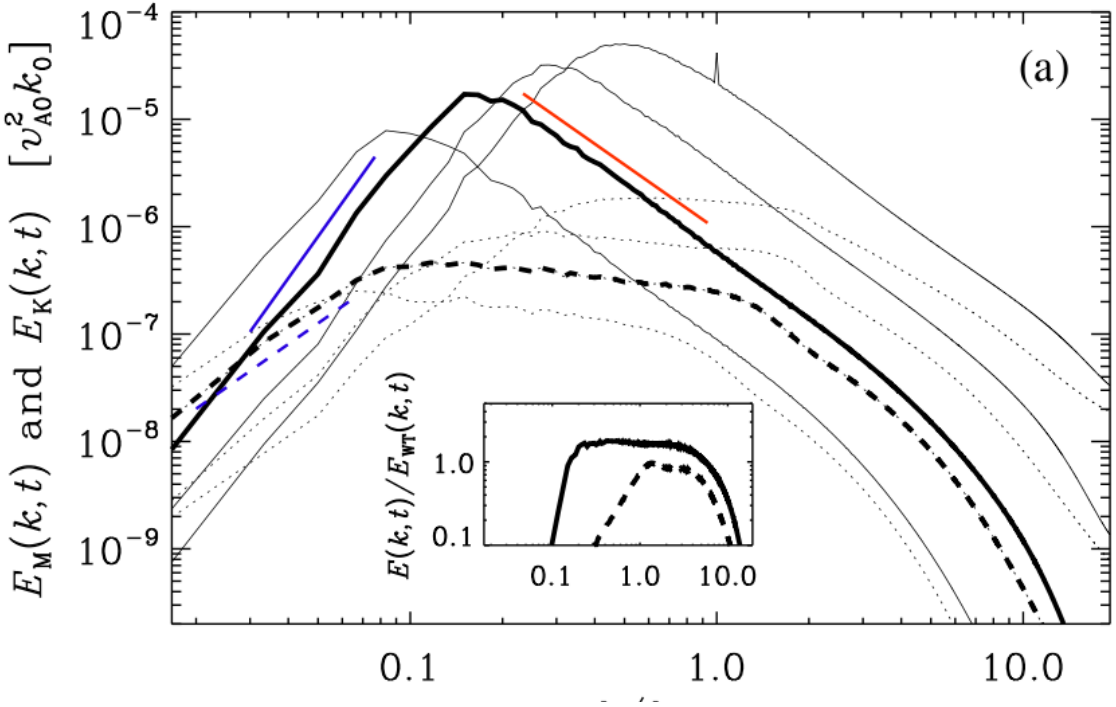
$$E \propto t^{-2/3}$$

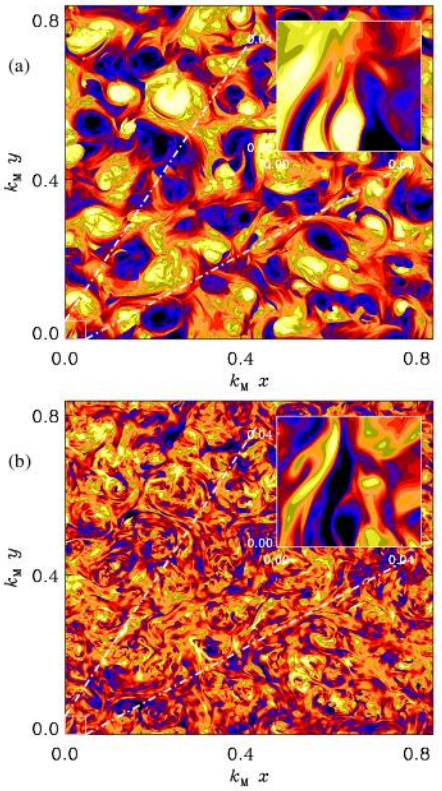
$$L \propto t^{+2/3}$$

Nonhelical Inverse Transfer of a Decaying Turbulent Magnetic Field

Axel Brandenburg,[1,2,*] Tina Kahniashvili,[3,4,5,†] and Alexander G. Tevzadze[6,‡]

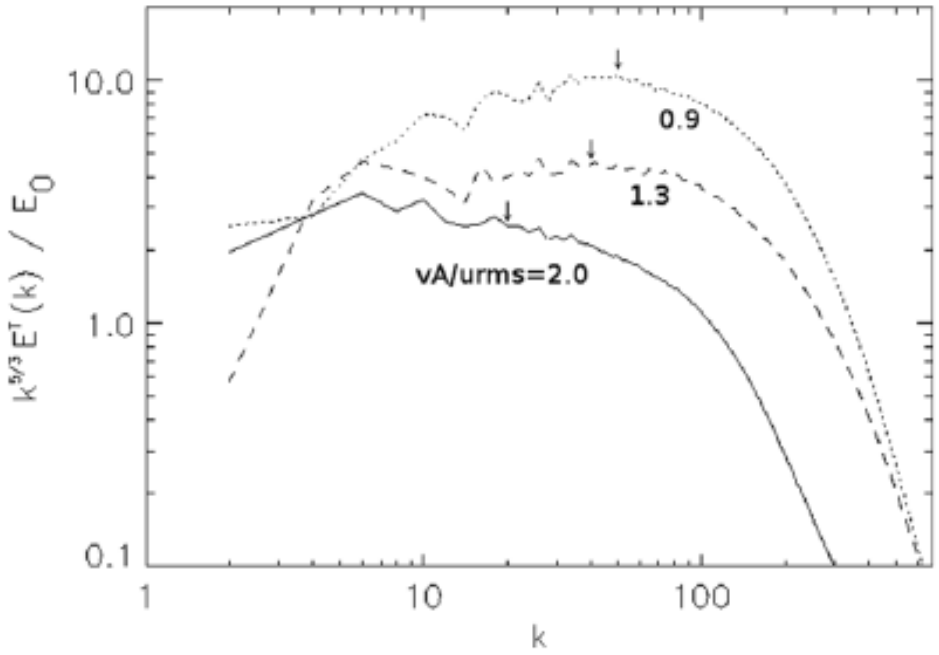
[1]*Nordita, KTH Royal Institute of Technology and Stockholm University, Roslagstullsbacken 23, 10691 Stockholm, Sweden*
[2]*Department of Astronomy, AlbaNova University Center, Stockholm University, 10691 Stockholm, Sweden*
[3]*The McWilliams Center for Cosmology and the Department of Physics, Carnegie Mellon University, 5000 Forbes Avenue, Pittsburgh, Pennsylvania 15213, USA*
[4]*Department of Physics, Laurentian University, Ramsey Lake Road, Sudbury, Ontario P3E 2C, Canada*
[5]*Abastumani Astrophysical Observatory, Ilia State University, 3-5 Cholokashvili Avenue, Tbilisi GE-0194, Georgia*
[6]*Faculty of Exact and Natural Sciences, Tbilisi State University, 1 Chavchavadze Avenue, Tbilisi 0128, Georgia*





$$E_{\rm WT}(k,t) = C_{\rm WT}(\epsilon v_A k_M)^{1/2} k^{-2}$$

Weak MHD turbulence, because B strong

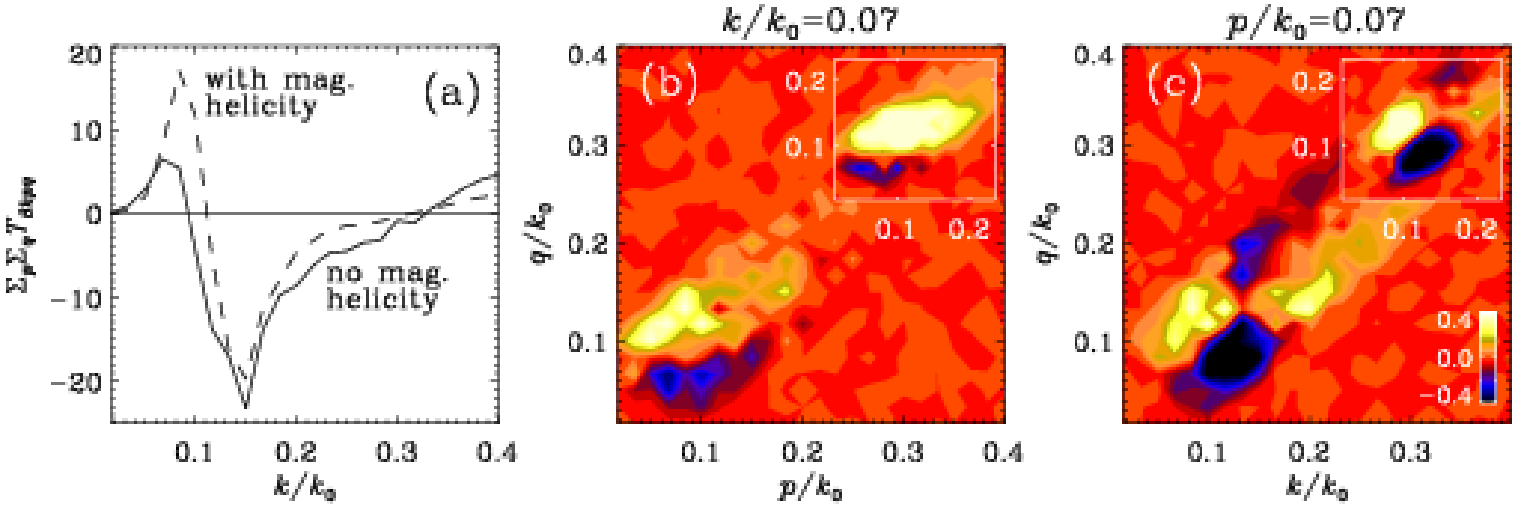


Lee, Brachet, Pouquet, Mininni, Rosenberg (2010)

	Iroshnikov–Kraichnan (isotropic, sub-equip.)	Strong turbulence (critically balanced)	Weak turbulence (wave turbulence)
$v_A/u_{rms} \sim \chi^{-1}$	<1	~ 1	>1
τ_{casc}	$\chi^{-2}\tau_A$ (with $k_\perp = k_\parallel$)	$\chi^{-1}\tau_A (= \tau_{NL})$	$\chi^{-2}\tau_A$
ϵ	$z_k^4 k / v_A$	$z_{k_\perp}^3 k_\perp$	$z_{k_\perp}^4 k_\perp^2 / v_A k_\parallel$
$k_\perp / k_\parallel$	1	$\propto k_\perp^{1/3}$	$\rightarrow \infty$
$E(k_\perp, k_\parallel)$	$(\epsilon v_A)^{1/2} k^{-3/2}$	$\epsilon^{2/3} k_\perp^{-5/3}$	$(\epsilon v_A k_\parallel)^{1/2} k_\perp^{-2}$

Inverse transfer similar to helical MHD

$$T_{kpq} = \left\langle \mathbf{J}^k \cdot \left(\mathbf{u}^p \times \mathbf{B}^q \right) \right\rangle = -\left\langle \mathbf{u}^p \cdot \left(\mathbf{J}^k \times \mathbf{B}^q \right) \right\rangle$$



Nonhelical gain
½ of helical case

Gain from SS ***B***
Mediated by LS ***u***

Kinetic gain
From ***B*** field

On Inverse Cascades and Primordial Magnetic Fields

P. Olesen[1]
The Niels Bohr Institute, University of Copenhagen
Blegdamsvej 17, DK-2100 Copenhagen Ø Denmark

Next we want to use the well known self-similarity property of the non-relativistic Navier-Stokes or MHD-equations,

$$\mathbf{x} \to l\mathbf{x}, \quad t \to l^{1-h}t, \quad \mathbf{v} \to l^h\ \mathbf{v}, \quad \nu \to l^{1+h}\nu, \quad \mathbf{B} \to l^h\ \mathbf{B}, \ \eta \to l^{1+h}\eta, \tag{5}$$

where ν is the kinetic and η is the Ohmic diffusion. Using the substitutions $\mathbf{x} = l\mathbf{x}'$ and $\mathbf{y} = l\mathbf{y}'$, we obtain from eqs. (2) and (5)

$$\begin{aligned} E(k/l, l^{1-h}t, Ll, K/l) &= l^4 \frac{2\pi k^2}{(2\pi)^3} \int_{2\pi/K}^{L} d^3x' d^3y'\ e^{i\mathbf{k}(\mathbf{x}'-\mathbf{y}')}\ <\mathbf{v}(l\mathbf{x}', l^{1-h}t)\ \mathbf{v}(l\mathbf{y}', l^{1-h}t)> \\ &= l^{4+2h}\ \tilde{E}(k, t, L, K). \end{aligned} \tag{6}$$

primordial magnetic fields, and for the effect of diffusion. In general, if the initial spectrum is k^α, then in the "inertial" range, for $\alpha > -3$ there is an inverse cascade, whereas for $\alpha < -3$ there is a forward cascade.

Does initial spectrum determine decay?

$$E_{\rm K}(k,t) \sim E_{\rm M}(k,t) \sim k^{\alpha}\psi(k^{(3+\alpha)/2}t). \qquad (4)$$

Integrating over k yields the decay law of the energies as

$$\mathcal{E}_{\rm K}(t) = \int E_{\rm K}(k,t)\,dk \sim \int k^{\alpha}\psi(k^{(3+\alpha)/2}t)\,dk. \qquad (5)$$

Introducing $\kappa = kt^q$ with $q = 2/(3+\alpha)$, we have

$$\mathcal{E}_{\rm K}(t) \sim t^p \int \kappa^{\alpha}\psi(\kappa)\,d\kappa, \qquad (6)$$

where $p = (1+\alpha)q$. The integral scales like $k_K \sim t^q$ with $q = 2/(3+\alpha)$. Several parameter combinations are given in Table II.

α	p	q
4	10/7	2/7
3	8/6	2/6
2	6/5	2/5
1	4/4	2/4
0	2/3	2/3

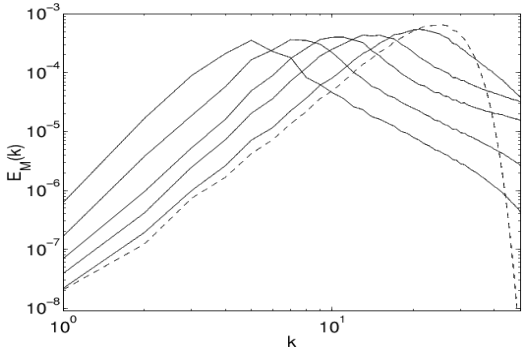
$$q = 1 - p/2$$

Rescaled spectra: self-similar

Alternative interpretation of Olesen's scaling relation

Christensson et al.
(2001, PRE 64, 056405)

Initial slope
$E \sim k^4$



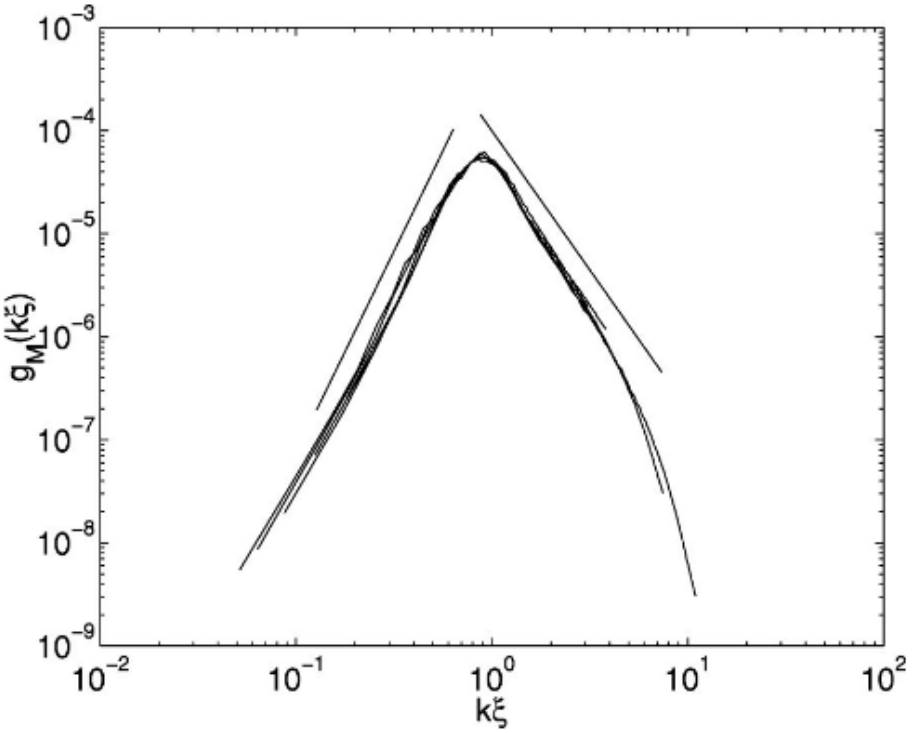


FIG. 2. The magnetic scaling function $g_M(k\xi)$ described in the text, Eq. (13), versus $k\xi$. The straight lines indicate the power laws $\propto (k\xi)^{4.0}$ and $\propto (k\xi)^{-2.5}$, respectively.

$$E_M(k,t) = \xi(t)^{-q} g_M(k\xi). \qquad (13)$$

Revised interpretation

$$E(k,t) = \xi\phi(k\xi)$$

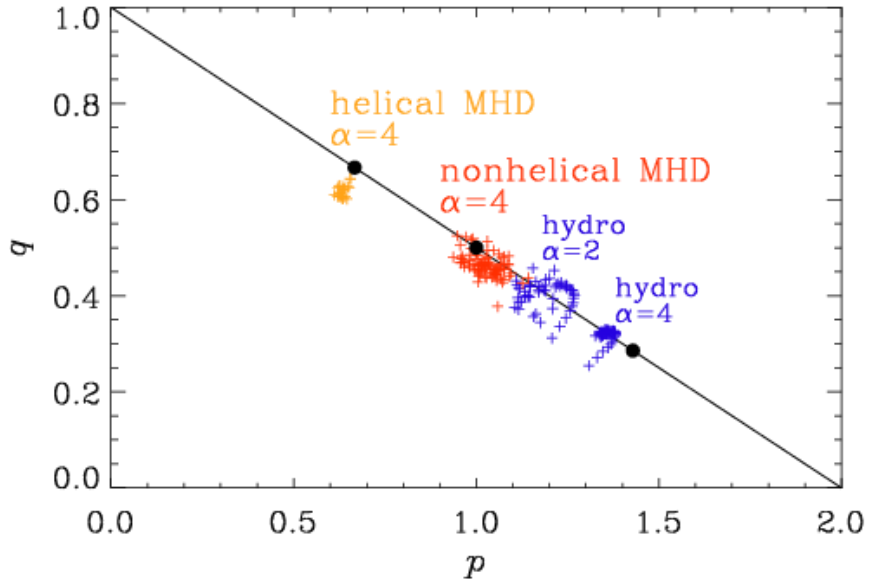
with integral scale ξ

$$\xi = \xi(t) \propto t^q$$

and q determined by physics

$$\beta = -3 + 2/q$$

from dimensional arguments



$$\mathcal{L} = \int \boldsymbol{r}^2 \langle \boldsymbol{u}(\boldsymbol{x}) \cdot \boldsymbol{u}(\boldsymbol{x}+\boldsymbol{r}) \rangle \, d\boldsymbol{r} \propto \ell^5 u_\ell^2,$$

β	p	q	physics
4	10/7	2/7	$\int u^2 r^4 \, \mathrm{d}r = \mathcal{L} = \mathrm{const} \sim \ell^7 \tau^{-2}$ (Loitsiankii)
3	8/6	2/6	
2	6/5	2/5	$\int u^2 r \, \mathrm{d}r = \mathcal{C} = \mathrm{const} \sim \ell^5 \tau^{-2}$ (Saffman)
1	4/4	2/4	$\langle \boldsymbol{A}_{\mathrm{2D}}^2 \rangle = \mathrm{const} \sim \ell^4 \tau^{-2}$ (to be confirmed)
0	2/3	2/3	$\langle \boldsymbol{A} \cdot \boldsymbol{B} \rangle = \mathrm{const} \sim \ell^3 \tau^{-2}$ (Biskamp & Müller)

Scaling relations

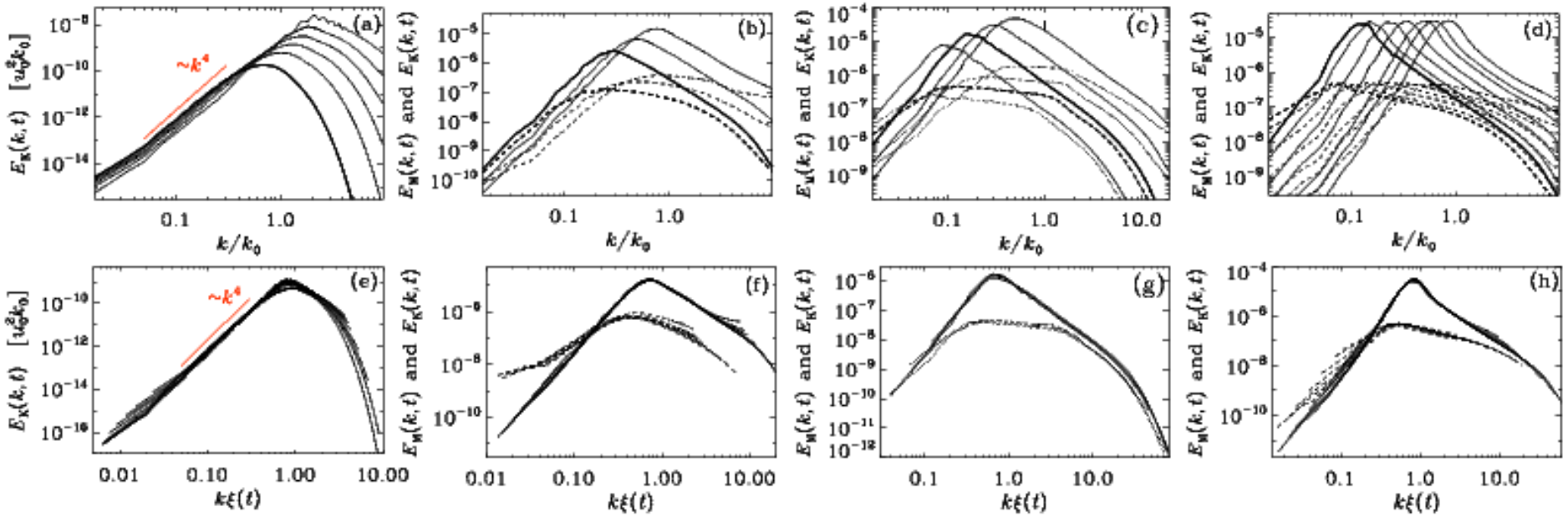
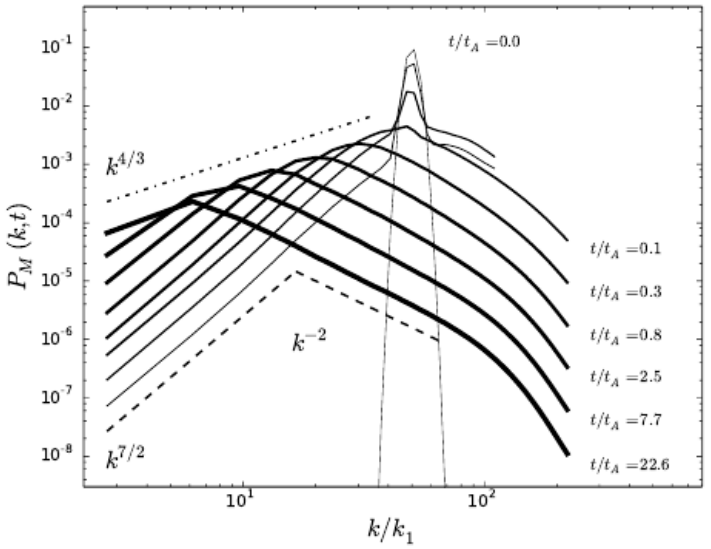


FIG. 1: Kinetic energy spectra in a hydrodynamic simulation (a), compared with magnetic (solid) and kinetic (dashed) energy spectra in a hydromagnetic simulation without helicity (b) and (c), and with (d). Panels (e)–(h) show the corresponding collapsed spectra obtained by using $\beta_{\rm M} = 3$ (e), $\beta_{\rm M} = 2$ (f), $\beta = 1$ (g), and $\beta = 0$ (h). In (f) we used $\beta_{\rm K} = 1 \neq \beta_{\rm M}$.

Conclusions

- Helicity slows down decay
- Large scale energy increases
- Nonhelical inverse transfer
- Revised interpretation to Olesen
- Self-similar spectra
- Confirmed now by others (Berera & Linkmann 2015 Zrake 2015)