

An Introduction to Planetary and Stellar Orbits

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Background

The theory of planetary and stellar orbits (also referred to as celestial mechanics) has its roots back in medieval times when Johannes Kepler used Tycho Brahe's observations of planet motions to show that the planets move on ellipses around the sun and that the square of the orbital period is proportional to the cube of the major axis. Over the past few decades however celestial mechanics has gained significantly in importance. With the advent of interplanetary space flight celestial mechanics was no longer something passive (purely diagnostic), but became something active in the sense that one was now able to choose what orbits to fly in. Furthermore, the detection of some thirty exoplanets (planets around other stars) has been possible using celestial mechanics by observing changes in the proper motions of the stars alone. With the ever increasing precision of such measurements the requirements on orbit calculations are bound to increase too.

The present notes follow quite closely the excellent book by AE Roy *Orbital Motion* (Institute of Physics Publishing, Bristol 1988). That is also the reason why, except for Chapter 1, the numbering of all sections and equations is consistent with that book. However, because the book covers much more material than we cover in this course, many chapters and sections are left out, and only three chapters are currently included: Chapters 4, 5, and 11.

We shall mostly be concerned with orbits in the solar system. For future reference we give here some relevant parameters of some celestial bodies in the solar system. (m is mass, a is semi-major axis, and T is orbital period.)

Body	M [kg]	a [10^6 km]	a [AU]	T [yr]
Earth	6.0×10^{24}	149.5	1	1
Mars	6.4×10^{23}	227.8	1.52	1.88
Jupiter	1.9×10^{27}	778	5.20	11.86
Sun	2.0×10^{30}	0	0	–

The use of greek symbols is quite common. To help familiarising with these symbols I give below a table of frequently used symbols, how they are pronounced, and what their meanings are. This table is also useful because sometimes the same symbols are used for different things. (It's all consistent with Roy's book.)

symbol	name	meaning	chapter
θ	theta	polar angle	4
μ	mu	abbreviation for $G(m_1 + m_2)$	4, 11
		fractional mass, $m_2/(m_1 + m_2)$	5
ξ	xi	x coordinate of 3rd body	5.10.1
		x coordinate in linearised equation	5.10.4
η	eta	y coordinate of 3rd body	5.10.1
		y coordinate in linearised equation	5.10.4
ζ	zeta	z coordinate of 3rd body	5.10.1
		z coordinate in linearised equation	5.10.4
λ	lambda	characteristic value (eigenvalue), as in $e^{\lambda t}$	5.10.4

Chapter 1

Mass, Momentum, and all that

In this introductory chapter we shall briefly review some basic concepts of mechanics used later on. Most of this can easily be verified with simple experiments, and corresponding sketches will be given here.

1.1 Mass, weight and acceleration

Mass is what is usually called ‘weight’, although the two are not the same. Weight is really what is indicated on the scales. The more massive a body is, the more the spring in the scales will expand. If the same experiment is carried out on the moon however, the weight will be six times less, even though the mass is still the same! The equation that relates the two is

$$F = mg \quad (1.1)$$

where F is the force pulling on the scales (ie the weight of the body), m is the mass of the body, and g is the gravitational acceleration. It’s called acceleration, because this quantity gives the rate at which the velocity of a freely falling body increases, so

$$\frac{dv}{dt} \equiv \text{acceleration} = g. \quad (1.2)$$

If you let a body drop from the Claremont tower, you will find that after each second its velocity has increased by approximately 9.81 meters per second. Thus, on the earth surface $g \approx 9.81 \text{ m s}^{-1}$, but the exact value depends somewhat on latitude, because the earth is not exactly round. (On the moon g is roughly six times less.)

Multiplying eq. (1.2) by m gives the force, ie

$$m \frac{dv}{dt} \equiv \text{force} = F. \quad (1.3)$$

If there are other forces (eg from other bodies) pulling on the body then the motion of that body will be governed by the sum of all these forces, ie

$$m \frac{dv}{dt} = \sum_i F_i. \quad (1.4)$$

1.2 Free fall and bullet trajectories

The motion of a body, as described by eq. (1.2), is independent of its mass. (If frictional drag were absent a feather would fall onto the floor in the same time as a piece of metal.) Since g is constant (1.2) can be integrated to give

$$v = gt, \quad (1.5)$$

where we have used that $v(0) = 0$. The position of the body z is given by $dz/dt = v$, and integration of t gives $\frac{1}{2}t^2$, so

$$z = z_0 - \frac{1}{2}gt^2, \quad (1.6)$$

where we have used the fact that gravity is directed downwards, but our z coordinate increases upwards.

When applying this to the problem of bullet trajectories the equations are the same, but the initial conditions are different. It is also now important to use the vectorial nature of gravity, velocity, and position. Thus, we have $d\mathbf{v}/dt = \mathbf{g}$ and $d\mathbf{r}/dt = \mathbf{v}$ using

$$\mathbf{g} = \begin{pmatrix} 0 \\ 0 \\ -g \end{pmatrix}, \quad \mathbf{v}_0 = \begin{pmatrix} v_{x0} \\ 0 \\ v_{z0} \end{pmatrix}, \quad \mathbf{r}_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \quad (1.7)$$

The solution is then

$$\mathbf{r} = \begin{pmatrix} v_{x0}t \\ 0 \\ v_{z0}t - \frac{1}{2}gt^2 \end{pmatrix}, \quad (1.8)$$

which corresponds to a *parabolic* trajectory.

1.3 Energy and energy conservation

There are different forms of energy. Here we will mostly be concerned with kinetic and potential energies, but there are also thermal, electric and magnetic energies, for example. Energy conservation means that in a closed system the sum of all energies is constant. If you drop a stone, for example, there was initially only potential energy (which is equal to mgh , where h is height), but then as the stone falls its potential energy decreases and the stone gains kinetic energy ($\frac{1}{2}mv^2$). In that case energy conservation means

$$\frac{1}{2}mv^2 + mgh = \text{const.} \quad (1.9)$$

(You may not notice it, but when the stone hits the floor kinetic energy will be transformed into heat, ie thermal energy.) Applying (1.9) to the case of a pendulum we may set the constant equal to mgh_0 , where h_0 is the initial height. This allows us to calculate the maximum speed of the pendulum when it goes through the minimum: $v_{\max} = \sqrt{2gh_0}$.

1.4 Actio and reactio

The principle of *actio and reactio* is quite important, but it may seem at first somewhat strange. The point is that bodies don't exert forces upon themselves, but forces always originate because of somebody else. If A pulls on B with the force $\mathbf{F}$, then B must necessarily also pull on A, but with the opposite force $-\mathbf{F}$. This is the principle of *actio and reactio*. If B lets go, then A will be unable to continue pulling on B.

This principle is also important in collision problems (next section). When a body hits another body then the speeds of both bodies change respectively by the amounts $\Delta\mathbf{v}_1$ and $\Delta\mathbf{v}_2$ during the short time span Δt . Now because of *actio and reactio*, $\mathbf{F}_1 = -\mathbf{F}_2$, and because of (1.3), we have

$$m_1 \frac{\Delta\mathbf{v}_1}{\Delta t} = -m_2 \frac{\Delta\mathbf{v}_2}{\Delta t}, \quad (1.10)$$

so

$$m_1 \Delta\mathbf{v}_1 + m_2 \Delta\mathbf{v}_2 = 0, \quad (1.11)$$

or, denoting the velocities after impact by capital letters,

$$m_1 v_1 + m_2 v_2 = m_1 V_1 + m_2 V_2. \quad (1.12)$$

1.5 Momentum and momentum conservation

The momentum of a body of mass m and velocity $\mathbf{v}$ is defined as $m\mathbf{v}$. Since the mass of a body remains unchanged, its momentum evolution is governed by equation (1.3). If there is no external force acting on the body, its momentum $m\mathbf{v}$ will remain constant. When two bodies collide, the total momentum, $m_1\mathbf{v}_1 + m_2\mathbf{v}_2$, as we have seen above, will be conserved. This is well illustrated by the colliding pendulum experiment; see figure 1.1.

To describe general collision problems completely we need to invoke both energy and momentum conservation, because we have two unknowns (the two velocities after impact), so we need two equations; see the box below.

Denoting the initial velocities by lower case v and the final velocities by upper case V the law of momentum conservation gives

$$m_1v_1 + m_2v_2 = m_1V_1 + m_2V_2, \quad (1.13)$$

and the law of energy conservation gives

$$\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 = \frac{1}{2}m_1V_1^2 + \frac{1}{2}m_2V_2^2. \quad (1.14)$$

We know the initial velocities, v_1 and v_2 , and the two masses, m_1 and m_2 . We have two unknowns, V_1 and V_2 , with two equations, so we can solve this. When doing the analysis one would first eliminate V_2 . It is useful to work with the ratios $\mu = m_1/m_2$ and $r = V_1/v_1$. In the special case where the second body is initially at rest we have

$$\frac{V_1}{v_1} = \frac{m_1 - m_2}{m_1 + m_2}, \quad \frac{V_2}{v_1} = \frac{2m_1}{m_1 + m_2}. \quad (1.15)$$

In the more general case of nonvanishing v_2 the result is

$$V_1 = \frac{2m_2v_2 + v_1(m_1 - m_2)}{m_1 + m_2}, \quad V_2 = \frac{2m_1v_1 + v_2(m_2 - m_1)}{m_1 + m_2}. \quad (1.16)$$

1.6 Circular motion

In order to force a body on a circular orbit one has to prevent it from flying straight; see figure 1.2. To do this one needs to accelerate the body constantly towards the centre. We denote the orbital speed by V . After a time Δt the body would fly straight over the distance $V\Delta t$. To compensate for the departure from the curved trajectory the body must be accelerated radially inwards with the acceleration $\mathbf{a}$ such that it travels a radial distance $\frac{1}{2}at^2$.

We now want to work out the magnitude of the acceleration a needed to keep a body with orbital speed V in a circular orbit of radius r . In terms of cartesian xy coordinates a circle is given by

$$x^2 + y^2 = r^2 \quad (1.17)$$

In this expression we identify y with $V\Delta t$ and x with $r - \frac{1}{2}a\Delta t^2$, so

$$(r - \frac{1}{2}a\Delta t^2)^2 + (V\Delta t)^2 = r^2. \quad (1.18)$$

This gives

$$r^2 - ra\Delta t^2 + \frac{1}{4}a^2\Delta t^4 + V^2\Delta t^2 = r^2, \quad (1.19)$$

or, after cancelling some terms, $-ra + \frac{1}{4}a^2\Delta t^2 + V^2 = 0$, but in the limit $\Delta t \rightarrow 0$ (continuous corrections) we have $-ra + V^2 = 0$, or

$$a = V^2/r \quad (1.20)$$

This is the expression for the acceleration needed to keep the body in orbit. The corresponding (inward pointing) force mV^2/r is also called *centripetal* force, whilst the outwards acceleration is called centrifugal acceleration.

1.7 Newton's law

Celestial bodies are rarely on a precisely circular orbit, but more often on a elliptic orbit. Nevertheless, in the solar system the *eccentricities* (explained later) of many orbits is small, so let's consider simple circular orbits first.

Kepler found that the square of the orbital period is proportional to the cube of the orbital radius, so $T^2 \sim r^3$. (In order to determine T one just needs a calendar to measure the time interval in which a planet has done one revolution. In the next section we show how distances to other planets can be measured relative to the earth's orbital radius.) Let's see whether we can infer from Kepler's law, $T^2 \sim r^3$, the radial dependence of the gravitational force $F(r)$ needed to keep the body in orbit. We therefore identify the centripetal force with the gravity force $F(r)$, so

$$F(r) = m \frac{V^2}{r}. \quad (1.21)$$

The circumference of the orbit is $2\pi r$, so the orbital period is $T = 2\pi r/V$, and therefore $V = 2\pi r/T$. Thus,

$$F(r) = m \frac{4\pi^2 r}{T^2}. \quad (1.22)$$

In order that Kepler's law, $T^2 \sim r^3$, is satisfied, we have to have

$$F(r) = \frac{\text{const}}{r^2}. \quad (1.23)$$

This is Newton's law of gravity. Newton also showed that the force must be proportional to the product of the central mass M and the mass m of the orbiter, so

$$F = G \frac{Mm}{r^2}, \quad (1.24)$$

where G is a universal constant. Thus, in the special case of circular orbits Kepler's 3rd law for the rotation period takes the form $T = 2\pi\sqrt{r^3/GM}$.

1.8 The distance to Mars

In order to proof Kepler's law and to determine the numerical value of the gravitational constant G one needs to know the distances to the various bodies. This is possible by measuring exactly the planets' motions on the sky. A curious observation is that around the time the planets are brightest (closest to the earth) they describe a loop; see figure 1.3. figure 1.4 shows the opposition loop of Mars in the year 1999.

By measuring the speed of a planet on the sky (let's use Mars here, although this technique works for other planets too) during its backwards motion in the loop we can calculate the radius of the orbit of the planet relative to that of the earth; see figure 1.5. After a time Δt Earth and Mars have moved by the linear distance d_1 and d_2 , respectively, where

$$d_1 = 2\pi r_1 \frac{\Delta t}{T_1}, \quad d_2 = 2\pi r_2 \frac{\Delta t}{T_2}. \quad (1.25)$$

So, relative to Mars the Earth has advanced by $d_1 - d_2$ whilst their relative distance is $r_2 - r_1$, so Mars will move backwards in the sky by the angle

$$\Delta\phi = \frac{d_1 - d_2}{r_2 - r_1}. \quad (1.26)$$

Moving $r_2 - r_1$ to the left hand side, $\Delta\phi$ to the right hand side, and using (1.25) we have

$$r_2 - r_1 = 2\pi \frac{\Delta t}{\Delta\phi} \left(\frac{r_1}{T_1} - \frac{r_2}{T_2} \right) \quad (1.27)$$

or, collecting the factors of r_1 and r_2 ,

$$\frac{r_2}{r_1} = \frac{1 + \frac{2\pi\Delta t}{T_1\Delta\phi}}{1 + \frac{2\pi\Delta t}{T_2\Delta\phi}} \quad (1.28)$$

From figure 1.4 we see in half a month ($\Delta t = 1/24$ yr) Mars has moved backwards by 40 minutes. (Astronomers measure the azimuthal angle – from west to east – not in degrees or radians, but in hours, so 24 hours corresponds to 2π .) So for a $\Delta\phi$ of 40 minutes we have $\Delta\phi = 2\pi \times 40/(60 \times 24)$, and with $T_1 = 1$ yr and $T_2 \approx 2$ yr we have

$$\frac{r_2}{r_1} \approx \frac{1 + \frac{1/24}{2/(3 \times 24)}}{1 + \frac{1/24}{4/(3 \times 24)}} = \frac{1 + 3/2}{1 + 3/4} = \frac{10/4}{7/4} = 10/7, \quad (1.29)$$

which, given the accuracy of this calculation, is remarkably close to the exact value of 1.5.

1.9 Analogy between linear and circular motions

There is a close analogy between linear and circular motions. Distance corresponds to angle, whilst velocity and acceleration correspond to angular velocity and angular acceleration, respectively. Although we haven't really discussed the counterparts to mass and force and other quantities at all, we mention them now with their definitions in tabular form; see Table 1.1.

distance x	angle ϕ
velocity $v = \dot{x} = \ddot{x}$	angular velocity $\omega = \dot{\phi} = \ddot{\phi}$
acceleration $a = \dot{v}$	angular acceleration $\alpha = \dot{\omega}$
mass $m = \sum_i \Delta m_i$	moment of inertia $I = \sum_i \Delta m_i r_i^2$
force $F = ma$	torque $T = I\alpha$
kinetic energy $E_{\text{kin}} = \frac{1}{2}mv^2$	rotational energy $E_{\text{rot}} = \frac{1}{2}I\omega^2$
momentum $p = mv$	angular momentum $L = I\omega$

With those rather informal preparations we will now launch into the more mathematical discussions of Kepler's laws, allowing for the more general non-circular orbits.

Figure 1.1: Illustration of momentum conservation: the ball on the left hits the other balls with a certain velocity v , and so its momentum is mv , where m is the mass of each ball. When it hits, the push is transmitted to all other balls and, since momentum is conserved, the right most ball moves to the right at the same speed.

Figure 1.2: In order to force a body on a circular orbit one has to prevent it from flying straight. This requires an acceleration a .

Figure 1.3: Loop in the motion of Mars during opposition as seen from Earth relative to the background stars.

Figure 1.4: Loop in the motion of Mars between January and August 1999. The numbers give the month.

Figure 1.5: Positions of Earth and Mars at time Δt after opposition. The opening angle of the triangle gives the backward motion.

Chapter 4

The Two-Body Problem

4.4 The Solution to the Two-Body Problem

In the following we consider two masses, m_1 and m_2 , so their mutual gravitational attraction force is $F = Gm_1m_2/r^2$; see (1.24). We denote the position vectors of the two bodies by $\mathbf{r}_1$ and $\mathbf{r}_2$, so their mutual distance is $r = |\mathbf{r}_2 - \mathbf{r}_1|$, the velocities of both bodies are $\dot{\mathbf{r}}_1$ and $\dot{\mathbf{r}}_2$, and their accelerations are $\ddot{\mathbf{r}}_1$ and $\ddot{\mathbf{r}}_2$. The equations of motion of both bodies are then

$$m_1\ddot{\mathbf{r}}_1 = G\frac{m_1m_2}{r^3}\mathbf{r}, \quad (4.6)$$

$$m_2\ddot{\mathbf{r}}_2 = -G\frac{m_1m_2}{r^3}\mathbf{r}, \quad (4.7)$$

where $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$ is a vector pointing from m_1 to m_2 . Adding the two equations (4.6) and (4.7) gives

$$m_1\ddot{\mathbf{r}}_1 + m_2\ddot{\mathbf{r}}_2 = 0,$$

giving the two integrals

$$m_1\dot{\mathbf{r}}_1 + m_2\dot{\mathbf{r}}_2 = \mathbf{a}, \quad (4.8)$$

$$m_1\mathbf{r}_1 + m_2\mathbf{r}_2 = \mathbf{a}t + \mathbf{b}, \quad (4.9)$$

where $\mathbf{a}$ and $\mathbf{b}$ are integration constants. (Our first 6 out of a total of 12 integration constants!)

Introducing the abbreviations

$$M\mathbf{R} = m_1\mathbf{r}_1 + m_2\mathbf{r}_2$$

and

$$M = m_1 + m_2$$

we can rewrite (4.8) and (4.9) as

$$M\dot{\mathbf{R}} = \mathbf{a}, \quad M\mathbf{R} = \mathbf{a}t + \mathbf{b},$$

which shows that the position of the centre of mass,

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}$$

moves with constant velocity, $\dot{\mathbf{R}} = \mathbf{a}/M$.

Dividing (4.6) by m_1 and (4.7) by m_2 gives

$$\ddot{\mathbf{r}}_1 = Gm_2 \frac{\mathbf{r}}{r^3} \quad (4.10)$$

and

$$\ddot{\mathbf{r}}_2 = -Gm_1 \frac{\mathbf{r}}{r^3}. \quad (4.11)$$

Subtracting the two equations (4.11) from (4.10) gives

$$\frac{d^2}{dt^2}(\mathbf{r}_2 - \mathbf{r}_1) = -G(m_1 + m_2) \frac{\mathbf{r}}{r^3}$$

or

$$\ddot{\mathbf{r}} + G(m_1 + m_2) \frac{\mathbf{r}}{r^3} = 0.$$

We introduce the abbreviation $\mu = G(m_1 + m_2)$ to write this as

$$\ddot{\mathbf{r}} + \mu \frac{\mathbf{r}}{r^3} = 0. \quad (4.12)$$

We now perform two different vector manipulations with Eq. (4.12): taking the vector product with $\mathbf{r}$ and taking the dot product with $\dot{\mathbf{r}}$ and integrate. So taking first the vector product of $\mathbf{r}$ with Eq. (4.12) we obtain

$$\mathbf{r} \times \ddot{\mathbf{r}} = 0.$$

Noting that

$$\frac{d}{dt}(\mathbf{r} \times \dot{\mathbf{r}}) = \dot{\mathbf{r}} \times \dot{\mathbf{r}} + \mathbf{r} \times \ddot{\mathbf{r}} = \mathbf{r} \times \ddot{\mathbf{r}},$$

we see that integrating once yields

$$\mathbf{r} \times \dot{\mathbf{r}} = \mathbf{h}, \quad (4.13)$$

where $\mathbf{h}$ is another integration constant, pointing in the same direction for all t . (We have thus a total of 9 integration constants now.) The motion of one body about the other lies in a plane defined by the direction of $\mathbf{h}$. The area per unit time covered by $\mathbf{r}$ during the orbit is constant; see Fig. 4.1.

In polar coordinates (r, θ) we have

$$\begin{aligned} \mathbf{r} &= \hat{\mathbf{I}}r \\ \dot{\mathbf{r}} &= \dot{\hat{\mathbf{I}}}r + \hat{\mathbf{I}}\dot{r} = \hat{\mathbf{J}}\dot{\theta}r + \hat{\mathbf{I}}\dot{r} \end{aligned} \quad (4.14)$$

where $\hat{\mathbf{I}}$ and $\hat{\mathbf{J}}$ are unit vectors in the r and θ directions. In deriving (4.14) we have made use of the fact that $\dot{\hat{\mathbf{I}}} = \hat{\mathbf{J}}\dot{\theta}$; see Fig. 4.2. Thus (4.13) becomes

$$\mathbf{h} = \hat{\mathbf{I}}r \times (\hat{\mathbf{J}}\dot{\theta}r + \hat{\mathbf{I}}\dot{r}) = r^2\dot{\theta}\hat{\mathbf{K}},$$

where $\hat{\mathbf{K}} = \hat{\mathbf{I}} \times \hat{\mathbf{J}}$. Thus,

$$h = r^2\dot{\theta}. \quad (4.15)$$

Now we come to the last manipulation on (4.12); this time we take the dot product of (4.12) with $\dot{\mathbf{r}}$:

$$\dot{\mathbf{r}} \cdot \ddot{\mathbf{r}} + \mu \frac{\dot{\mathbf{r}} \cdot \mathbf{r}}{r^3} = 0.$$

Integrating this gives

$$\frac{1}{2}\dot{\mathbf{r}}^2 - \frac{\mu}{r} = C, \quad (*)$$

where C is our 10th integration constant. The first term is straightforward, because

$$\frac{d}{dt} \left(\frac{1}{2}\dot{\mathbf{r}}^2 \right) = \dot{\mathbf{r}} \cdot \ddot{\mathbf{r}}.$$

As for the second term we write

$$\frac{d}{dt} \left(\frac{1}{r} \right) = -\frac{\dot{r}}{r^2}.$$

However, using (4.14), and since $\hat{\mathbf{I}} \cdot \hat{\mathbf{J}} = 0$, we see that $\dot{\mathbf{r}} \cdot \mathbf{r} = \dot{r}r$. Therefore

$$\frac{d}{dt} \left(\frac{1}{r} \right) = -\frac{\dot{\mathbf{r}} \cdot \mathbf{r}}{r^3},$$

which verifies the second term in (*) above. Since $\dot{\mathbf{r}}^2 = v^2$ is the velocity squared, we can also write

$$\frac{1}{2}v^2 - \frac{\mu}{r} = C, \quad (4.16)$$

which describes energy conservation of the system. (The first term on the left hand side is related to the kinetic energy whilst the second term on the left hand side is related to potential energy. This equation also shows that the larger r is, the smaller is μ/r , and therefore the smaller is v . In other words, *at large distances the velocities are smaller*.)

In order to write Eq. (4.12) in polar coordinates we need to calculate first $\ddot{\mathbf{r}}$:

$$\ddot{\mathbf{r}} = \frac{d}{dt} (\hat{\mathbf{J}}\dot{\theta}r + \hat{\mathbf{I}}\dot{r}) = \dot{\hat{\mathbf{J}}}\dot{\theta}r + \hat{\mathbf{J}}\ddot{\theta}r + \hat{\mathbf{J}}\dot{\theta}\dot{r} + \dot{\hat{\mathbf{I}}}\dot{r} + \hat{\mathbf{I}}\ddot{r}.$$

We now need the derivative of $\hat{\mathbf{J}}$

$$\dot{\hat{\mathbf{J}}} = -\hat{\mathbf{I}}\dot{\theta},$$

see Fig. 4.2. Thus we have

$$\ddot{\mathbf{r}} = -\hat{\mathbf{I}}\dot{\theta}^2r + \hat{\mathbf{J}}\ddot{\theta}r + \hat{\mathbf{J}}\dot{\theta}\dot{r} + \hat{\mathbf{J}}\dot{\theta}\dot{r} + \hat{\mathbf{I}}\ddot{r} = \hat{\mathbf{I}}(\ddot{r} - \dot{\theta}^2r) + \hat{\mathbf{J}}(\ddot{\theta}r + 2\dot{\theta}\dot{r}).$$

This can be rewritten in the form

$$\ddot{\mathbf{r}} = \hat{\mathbf{I}}(\ddot{r} - \dot{\theta}^2r) + \hat{\mathbf{J}}\frac{1}{r}\frac{d}{dt}(\dot{\theta}r^2),$$

which can be verified by backwards differentiation. Now plugging this into (4.12) we have

$$\ddot{r} - \dot{\theta}^2r = -\frac{\mu}{r^2} \quad (4.17)$$

and

$$\frac{1}{r}\frac{d}{dt}(\dot{\theta}r^2) = 0. \quad (4.18)$$

The latter equation (4.18) yields the familiar integral

$$r^2\dot{\theta} = h, \quad (4.19)$$

which is just the scalar version of (4.13). The former equation (4.17) is more interesting and can be rewritten in the form

$$u'' + u = \mu/h^2$$

(see model solutions to exercise 1). The general solution is

$$u = \frac{\mu}{h^2} + A \cos(\theta - \omega), \quad (4.20)$$

where A and ω are two new integration constants. This gives now a total of 12 integration constants. Reintroducing r , Eq. (4.20) becomes

$$r = \frac{1}{\mu/h^2 + A \cos(\theta - \omega)} = \frac{h^2/\mu}{1 + (Ah^2/\mu) \cos(\theta - \omega)},$$

which has the form of the equation describing conic sections,

$$r = \frac{p}{1 + e \cos(\theta - \omega)}, \quad (4.21)$$

where $p = h^2/\mu$ and $e = Ah^2/\mu$ is the eccentricity. Plots of $r = r(f)$ for different values of e are given in Fig. 4.3.

Thus, the solution of the two-body problem is a conic section, and the type of section depends on the value of the eccentricity e :

- (i) for $e = 0$ the orbit is a circle,
- (ii) for $0 < e < 1$ the orbit is an ellipse,
- (iii) for $e = 1$ the orbit is a parabola,
- (iv) for $e > 1$ the orbit is a hyperbola

4.5 The Elliptic Orbit

In cartesian coordinates the *canonic* equation of an ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where a is the semimajor axis and b the semi-minor axis. To see the relation between this equation and (4.21) we define

$$x - x_0 = r \cos f, \quad y = r \sin f,$$

where $f = \theta - \omega$ is an abbreviation for the position angle from the perihelion. Unlike Fig. 4.3, the second body was in the origin (so $x = r \cos f$), here the centre of the ellipse is the origin. With these preparations we write (4.21) as

$$r + er \cos f = p,$$

or

$$r^2 = p^2 - 2per \cos f + e^2 (r \cos f)^2,$$

or

$$(x - x_0)^2 + y^2 = p^2 - 2pe(x - x_0) + e^2 (x - x_0)^2.$$

This can be written in the form

$$x^2(1 - e^2) + y^2 = p^2 - x_0^2(1 - e^2) + 2pex_0 + 2x[x_0(1 - e^2) - pe] \quad (**)$$

To bring this into the canonic form we choose x_0 such that the mixed term in angular brackets vanishes, ie

$$x_0 = \frac{pe}{1 - e^2}.$$

Then we have

$$x^2(1 - e^2) + y^2 = p^2 - \frac{(pe)^2}{1 - e^2} + 2\frac{(pe)^2}{1 - e^2} = p^2 + \frac{(pe)^2}{1 - e^2} = p^2 \frac{1}{1 - e^2}$$

Dividing this by $p^2/(1 - e^2)$ we obtain the canonic form with

$$a = \frac{p}{1 - e^2}, \quad \text{and} \quad \frac{b^2}{a^2} = 1 - e^2.$$

We now want to write down an equation for the orbital period T . We make use of the fact that the area covered by the position vector of the orbiter per unit time is $\frac{1}{2}h$. After the time T the area is $\frac{1}{2}hT$. But this has then to equal the area of the ellipse, which we know is πab . Hence we have

$$\frac{1}{2}hT = \pi ab.$$

Eliminating b , and multiplying by $2/T$, we have

$$h = \frac{2\pi}{T} a^2 \sqrt{1 - e^2}.$$

However, since $p = h^2/\mu = a(1 - e^2)$, we can eliminate h , so

$$\sqrt{\frac{a}{\mu}} \sqrt{1 - e^2} = \frac{2\pi}{T} a^2 \sqrt{1 - e^2},$$

or

$$T = 2\pi \sqrt{a^3/\mu}. \quad (4.26)$$

This equation allows us to compute the masses of celestial bodies, if the periods and semimajor axes of orbits around them are known.

4.5.2 Velocity in an elliptic orbit

Let V be the velocity of a body, then

$$V^2 \equiv \dot{\mathbf{r}}^2 = (\dot{r}\hat{\mathbf{I}} + r\dot{\theta}\hat{\mathbf{J}})^2 = \dot{r}^2 + r^2\dot{\theta}^2. \quad (4.32)$$

Differentiating (4.21) we have

$$\dot{r} = -\frac{p}{(1 + e \cos f)^2} (-e \sin f) \dot{\theta} = \frac{r^2}{p} e \sin f \dot{\theta}$$

Now since $r^2\dot{\theta} = h$, we have

$$\dot{r} = \frac{h}{p} e \sin f \quad (4.33)$$

On the other hand, from $r^2\dot{\theta} = h$ we also have

$$r\dot{\theta} = \frac{h}{r} = \frac{h}{p} (1 + e \cos f), \quad (4.34)$$

so we have for V^2

$$V^2 = \left(\frac{h}{p}\right)^2 [(e \sin f)^2 + 1 + 2e \cos f + (e \cos f)^2] = \left(\frac{h}{p}\right)^2 [(e^2 + 1 + 2e \cos f)],$$

or

$$V^2 = \left(\frac{h}{p}\right)^2 [(2 + 2e \cos f - (1 - e^2))], \quad (4.35)$$

or

$$V^2 = \left(\frac{h}{p}\right)^2 \left[2\frac{p}{r} - (1 - e^2)\right].$$

Using again $h^2/\mu = p = a(1 - e^2)$ we can eliminate p , so we have

$$V^2 = \left(\frac{h}{p}\right)^2 \left(2\frac{p}{r} - \frac{p}{a}\right) = \frac{h^2}{p} \left(\frac{2}{r} - \frac{1}{a}\right),$$

or

$$V^2 = \mu \left(\frac{2}{r} - \frac{1}{a}\right). \quad (4.36)$$

This equation can be used to calculate the orbital velocities at perihelion and aphelion, for example.

Equation (4.36) can be used to calculate the value of the semimajor axis:

$$a = \frac{\mu}{(2\mu/r) - V^2} \quad (4.37)$$

It is remarkable that a is the same once the values of V^2 and r are specified. Hence, the *direction* of the velocity does not matter; it only affects the eccentricity, but not the value of a . Furthermore, from Kepler's third law,

$$T = 2\pi \left(\frac{a^3}{\mu}\right)^{1/2}, \quad (4.38)$$

we can derive a relation between the period and the velocity,

$$T = 2\pi\mu \left(\frac{2\mu}{r} - V^2\right)^{-3/2}. \quad (4.38)$$

(This relation is used in one of the exercises.)

4.6 The Parabolic Orbit

For $e = 1$, Eq. (**) on p. 16 takes the form

$$y^2 = p^2 + 2px_0 - 2px,$$

which is indeed the equation of a parabola. If $x_0 = -p/2$ is chosen we have $y^2 = -2px$.

The velocity in a parabolic orbit can be calculated like before. The case $e = 1$ corresponds to $a \rightarrow \infty$, so Eq. (4.36) yields

$$V^2 = \mu \left(\frac{2}{r} - \frac{1}{a}\right) = \frac{2\mu}{r}. \quad (4.78)$$

There is an interesting relationship between this and the velocity V_c in a circular orbit, which is obtained by putting $a = r$ in Eq. (4.36):

$$V_c^2 = \mu \left(\frac{2}{r} - \frac{1}{r}\right) = \frac{\mu}{r}. \quad (4.79)$$

If the body in a circular orbit is given an impulse, so that its velocity becomes

$$V^2 = \frac{2\mu}{a}, \quad (4.80)$$

it will enter a parabolic orbit that will take it to infinity. It will reach infinity at zero velocity. It is seen from (4.79) and (4.80) that

$$\text{escape velocity} = \text{circular velocity} \times \sqrt{2}. \quad (4.81)$$

4.7 The Hyperbolic Orbit

We use Eq. (4.21) in the form

$$r = \frac{p}{1 + e \cos f}. \quad (4.88)$$

The canonic form of the equation of a hyperbola takes the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

where now

$$a = \frac{p}{e^2 - 1}, \quad \text{and} \quad \frac{b^2}{a^2} = e^2 - 1.$$

There is an angle $f = \pm\psi$ for which $r \rightarrow \infty$ giving the asymptote along which the body goes to infinity. The angle 2ψ between the two asymptotes of a hyperbolic orbit is reached when the denominator vanishes, ie

$$1 + e \cos \psi = 0, \quad (4.89)$$

so

$$\cos \psi = -\frac{1}{e}$$

Using the fact that $e = \sqrt{1 + (b/a)^2}$ and

$$1 + \tan^2 \psi = 1 + \frac{\sin^2 \psi}{\cos^2 \psi} = \frac{1}{\cos^2 \psi}$$

we have

$$\tan \psi = \pm b/a$$

for the half-angle between the two asymptotes.

4.7.1 Velocity in an hyperbolic orbit

Proceeding as in Sect. 4.5.2, we obtain an equation similar to (4.36), but with a different sign in front a ;

$$V^2 = \mu \left(\frac{2}{r} + \frac{1}{a} \right). \quad (4.91)$$

At perihelion, $f = 0$, we have $r = a(e^2 - 1)/(e + 1) = a(e - 1)$, so

$$V_p^2 = \frac{\mu}{a} \left(\frac{e + 1}{e - 1} \right). \quad (4.92)$$

Note that the body reaches infinity at finite speed.

4.9 Barycentric orbits

The orbits given above all refer to a frame of reference where one of the two bodies is at rest. We now want to derive the equations describing the motion in the frame of reference where the centre of mass is at rest. Thus, $m_1\mathbf{r}_1 + m_2\mathbf{r}_2 = 0$, or, in scalar notation where the distance from the centre of mass, R_1 and R_2 , are both positive numbers,

$$m_1 R_1 = m_2 R_2.$$

In addition we have

$$R_1 + R_2 = r$$

and

$$m_1 + m_2 = M.$$

We first eliminate R_2 , so

$$\frac{m_1}{m_2} R_1 = R_2 = r - R_1.$$

We then eliminate also $m_1 = M - m_2$ and multiply by m_2 , so

$$(M - m_2) R_1 = m_2 (r - R_1).$$

Here the $m_2 R_1$ term cancels on both sides, so

$$M R_1 = m_2 r,$$

or

$$R_1 = \frac{m_2}{M} r. \quad (*)$$

As similar law holds also for R_2 :

$$R_2 = \frac{m_1}{M} r.$$

We now want to express Kepler's second law, $r^2 \dot{\theta} = h$, in terms of (R_1, θ_1) and (R_2, θ_2) . Since the two bodies move about each other at the same rate as about their common centre of mass we have

$$\dot{\theta}_1 = \dot{\theta}_2 = \dot{\theta}.$$

Inserting this and Eq. (*) into Kepler's second law, $r^2 \dot{\theta} = h$, yields

$$\dot{R}_1 \left(\frac{M}{m_2} \right) \dot{\theta}_1 = h,$$

or, defining new values of h_1 and h_2 ,

$$R_1^2 \dot{\theta} = h_1$$

and

$$R_2^2 \dot{\theta} = h_2.$$

where R_1 and R_2 are the distances from the centre of mass, θ is the position angle (with the same rate of change for both of them), and h_1 and h_2 are given by

$$h_1 = \left(\frac{m_2}{M} \right)^2 r^2 \dot{\theta} = \left(\frac{m_2}{M} \right)^2 h,$$

and, similarly,

$$h_2 = \left(\frac{m_1}{M} \right)^2 h.$$

The barycentric orbits of m_1 and m_2 are therefore similar to each other and to their *relative* orbit. The values of the semimajor axis scales with mass just like the radius scales, ie

$$a_1 = \frac{m_2}{M} a.$$

and

$$a_2 = \frac{m_1}{M} a.$$

Application to binaries

Some binary stars are close enough so that their orbital parameters can be measured direction. The following consideration shows how this can be used to calculate the mass of the two components. Suppose you have measured a_1 , a_2 and the orbital period, T . From this you first calculate the total mass in terms of the solar mass, so

$$\frac{M}{M_\odot} = \left(\frac{a}{1 \text{ AU}} \right)^3 \left(\frac{T}{1 \text{ yr}} \right)^{-2},$$

where $a = a_1 + a_2$. (Example: $a_1 = 2 \text{ AU}$, $a_2 = 8 \text{ AU}$, $T = 20 \text{ yr}$, then $M = 2.5 M_\odot$.) One can then calculate their individual masses using the equations above: $m_1 = M a_2 / a = 2 M_\odot$ and $m_2 = M a_1 / a = 0.5 M_\odot$, where $a = 10 \text{ AU}$.

4.10 Classification of orbits with respect to the energy constant

We said earlier that the constant C in the equation of energy conservation,

$$\frac{1}{2} V^2 - \frac{\mu}{r} = C, \quad (4.110)$$

with $\mu = G(m_1 + m_2)$, is related to the total energy. The total energy is the sum of kinetic energy and potential energy, which in this case is

$$E = \frac{1}{2} m_1 V_1^2 + \frac{1}{2} m_2 V_2^2 - \frac{G m_1 m_2}{r}. \quad (*)$$

The velocities of the barycentric orbits scale with mass in the same way as the radii, so

$$V_1^2 = \dot{R}_1^2 + R_1^2 \dot{\theta}_1^2 = \left(\frac{m_2}{M} \right)^2 (\dot{r}^2 + r^2 \dot{\theta}^2) = \left(\frac{m_2}{M} \right)^2 V^2,$$

and similarly,

$$V_2^2 = \left(\frac{m_1}{M} \right)^2 V^2.$$

Inserting this into (*) yields

$$E = \frac{m_1 m_2}{m_1 + m_2} C.$$

There are three distinct cases:

- (i) for an ellipse, $V^2 = \mu(2/r - 1/a)$, and hence $C = -\mu/2a$
- (ii) for a parabola, $V^2 = 2\mu/r$, and hence $C = 0$
- (iii) for a hyperbola $V^2 = \mu(2/r + 1/a)$, and hence $C = +\mu/2a$

4.11 The orbit in space

Suppose $\mathbf{r}$ and $\dot{\mathbf{r}}$ are given in cartesian coordinates, we have

$$r^2 = x^2 + y^2 + z^2 \quad (4.116)$$

and

$$V^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2. \quad (4.117)$$

The components of the equation $\mathbf{r} \times \dot{\mathbf{r}} = \mathbf{h}$ then become

$$x\dot{y} - y\dot{x} = h_z$$

$$y\dot{z} - z\dot{y} = h_x$$

$$z\dot{x} - x\dot{z} = h_y$$

The equation

$$h^2 = \mu p$$

allows us then to calculate p , since $\mu = G(m_1 + m_2)$ is known. The type of conic section is then determined using (4.110), ie

$$\frac{1}{2}V^2 - \frac{\mu}{r} = C.$$

For an elliptic orbit the value of a is then obtained using

$$V^2 = \mu \left(\frac{2}{r} - \frac{1}{a} \right), \tag{4.36}$$

and likewise for other conic sections. The value of e is then obtained from the equation

$$p = a(1 - e^2).$$

Figure 4.1: The surface spanned by $\mathbf{r}$ and $\dot{\mathbf{r}}$ is given by the cross product of the two vectors, divided by two, ie $\frac{1}{2}|\mathbf{r} \times \dot{\mathbf{r}}|$.

Figure 4.2: The difference $\hat{\mathbf{I}}(\theta + \Delta\theta) - \hat{\mathbf{I}}(\theta)$ is equal to $\hat{\mathbf{J}}\Delta\theta$; thus $\Delta\hat{\mathbf{I}} = \hat{\mathbf{J}}\Delta\theta$ and so $d\hat{\mathbf{I}}/dt = \hat{\mathbf{J}}d\theta/dt$, or $\dot{\hat{\mathbf{I}}} = \hat{\mathbf{J}}\dot{\theta}$. On the other hand, the difference $\hat{\mathbf{J}}(\theta + \Delta\theta) - \hat{\mathbf{J}}(\theta)$ points in the *negative* $\hat{\mathbf{I}}$ direction, so it is equal to $-\hat{\mathbf{I}}\Delta\theta$; thus $\Delta\hat{\mathbf{J}} = -\hat{\mathbf{I}}\Delta\theta$ and so $d\hat{\mathbf{J}}/dt = -\hat{\mathbf{I}}d\theta/dt$, or $\dot{\hat{\mathbf{J}}} = -\hat{\mathbf{I}}\dot{\theta}$.

Figure 4.3: Plots of $r = r(\theta)$ for different values of e and $p = 1$. Here $x = r \cos f$ and $y = r \sin f$, where $f = \theta - \omega$.

Chapter 5

The Many-Body Problem

Already the three-body problem can no longer be solved in closed form. Only special cases can be solved. The known integrals concern only expressions involving sums over all particles.

5.2 The Equations of Motion in the Many-Body Problem

Define the vector that is directed from body i to body j as

$$\mathbf{r}_{ij} = \mathbf{R}_j - \mathbf{R}_i \quad (5.1)$$

The force on body i that results from all the other bodies $j \neq i$ is then

$$\mathbf{F}_i = G \sum_{j=1}^n \frac{m_i m_j}{r_{ij}^3} \mathbf{r}_{ij}, \quad i \neq j, \quad i = 1, 2, \dots, n.$$

Newton's law is then

$$m_i \ddot{\mathbf{R}}_i = G \sum_{j=1}^n \frac{m_i m_j}{r_{ij}^3} \mathbf{r}_{ij}, \quad i \neq j, \quad i = 1, 2, \dots, n. \quad (5.2)$$

Note that

$$\mathbf{r}_{ij} = -\mathbf{r}_{ji}. \quad (5.3)$$

5.3 The Ten Known Integrals and Their Meanings

The sum of all accelerations is

$$\sum_{i=1}^n m_i \ddot{\mathbf{R}}_i = 0.$$

This is because

$$\sum_{i=1}^n m_i \ddot{\mathbf{R}}_i = G \sum_{i=1}^n \sum_{j=1}^n \frac{m_i m_j}{r_{ij}^3} \mathbf{r}_{ij} = G \sum_{j=1}^n \sum_{i=1}^n \frac{m_j m_i}{r_{ji}^3} \mathbf{r}_{ji},$$

where we have just renamed the symbols $i \rightarrow j$ and $j \rightarrow i$. Reordering the symbols and interchanging the order of summation yields

$$\sum_{i=1}^n m_i \ddot{\mathbf{R}}_i = G \sum_{i=1}^n \sum_{j=1}^n \frac{m_i m_j}{r_{ij}^3} \mathbf{r}_{ji} = -G \sum_{i=1}^n \sum_{j=1}^n \frac{m_i m_j}{r_{ij}^3} \mathbf{r}_{ij}$$

where we have made use of (5.3). Thus $\sum_{i=1}^n m_i \ddot{\mathbf{R}}_i = 0$. Integrating once yields

$$\sum_{i=1}^n m_i \dot{\mathbf{R}}_i = \mathbf{a}, \quad (5.4)$$

which is related to the velocity of the centre of mass of all the bodies. In Chapter 4 we have derived this equation for the special case $n = 2$. Integrating a second time gives

$$\sum_{i=1}^n m_i \mathbf{R}_i = \mathbf{a}t + \mathbf{b}. \quad (5.5)$$

Defining, as before,

$$M\mathbf{R} = \sum_{i=1}^n m_i \mathbf{R}_i$$

and

$$M = \sum_{i=1}^n m_i$$

we have

$$\mathbf{R} = (\mathbf{a}t + \mathbf{b})/M, \quad (5.6)$$

which is the location of the centre of mass, and

$$\dot{\mathbf{R}} = \mathbf{a}/M, \quad (5.7)$$

which is the velocity of the centre of mass.

Again, we play similar tricks as before: take the cross product of (5.2) with $\mathbf{R}_i$ and the dot product of (5.2) with $\dot{\mathbf{R}}_i$ and sum over all i , so

$$\sum_{i=1}^n m_i \mathbf{R}_i \times \ddot{\mathbf{R}}_i = G \sum_{i=1}^n \sum_{j=1}^n \frac{m_i m_j}{r_{ij}^3} \mathbf{R}_i \times \mathbf{r}_{ij}, \quad j \neq i. \quad (5.8)$$

We can simplify the double sum by noting that for every pair (i, j) there is a corresponding pair (j, i) . Hence, we evaluate

$$\mathbf{R}_i \times \mathbf{r}_{ij} = \mathbf{R}_i \times (\mathbf{R}_j - \mathbf{R}_i) = \mathbf{R}_i \times \mathbf{R}_j$$

and

$$\mathbf{R}_j \times \mathbf{r}_{ji} = \mathbf{R}_j \times (\mathbf{R}_i - \mathbf{R}_j) = -\mathbf{R}_i \times \mathbf{R}_j.$$

Thus, to each pair (i, j) there is always a corresponding pair (j, i) , but it has the opposite sign, so they cancel. Therefore

$$\sum_{i=1}^n m_i \mathbf{R}_i \times \ddot{\mathbf{R}}_i = 0.$$

Integrating once gives

$$\sum_{i=1}^n m_i \mathbf{R}_i \times \dot{\mathbf{R}}_i = \mathbf{C}. \quad (5.9)$$

Until now we have 9 integration constants, three from $\mathbf{a}$, three from $\mathbf{b}$, and three from $\mathbf{C}$. The tenth one is obtained from the energy equation, which is generated by dotting (5.2) with $\dot{\mathbf{R}}_i$, so

$$\sum_{i=1}^n m_i \dot{\mathbf{R}}_i \cdot \ddot{\mathbf{R}}_i = G \sum_{i=1}^n \sum_{j=1}^n \frac{m_i m_j}{r_{ij}^3} \dot{\mathbf{R}}_i \cdot \mathbf{r}_{ij}, \quad j \neq i. \quad (5.10)$$

Now,

$$\dot{\mathbf{R}}_i \cdot \mathbf{r}_{ij} = \dot{\mathbf{R}}_i \cdot (\mathbf{R}_j - \mathbf{R}_i). \quad (5.11)$$

The same with interchanged coefficients involves

$$\dot{\mathbf{R}}_j \cdot \mathbf{r}_{ji} = -\dot{\mathbf{R}}_j \cdot (\mathbf{R}_i - \mathbf{R}_j) \quad (5.12)$$

Adding (5.11) and (5.12) we have

$$\dot{\mathbf{R}}_i \cdot \mathbf{r}_{ij} + \dot{\mathbf{R}}_j \cdot \mathbf{r}_{ji} = -(\dot{\mathbf{R}}_j - \dot{\mathbf{R}}_i) \cdot (\mathbf{R}_j - \mathbf{R}_i) = -\dot{\mathbf{r}}_{ij} \cdot \mathbf{r}_{ij}.$$

Since (5.11) and (5.12) are the same thing, just with different names, we use both and divide by two, so

$$\sum_{i=1}^n m_i \dot{\mathbf{R}}_i \cdot \ddot{\mathbf{R}}_i = -\frac{1}{2}G \sum_{i=1}^n \sum_{j=1}^n \frac{m_i m_j}{r_{ij}^3} \dot{\mathbf{r}}_{ij} \cdot \mathbf{r}_{ij}, \quad j \neq i.$$

or

$$\sum_{i=1}^n m_i \dot{\mathbf{R}}_i \cdot \ddot{\mathbf{R}}_i + \frac{1}{2}G \sum_{i=1}^n \sum_{j=1}^n \frac{m_i m_j}{r_{ij}^3} \dot{\mathbf{r}}_{ij} \cdot \mathbf{r}_{ij} = 0, \quad j \neq i.$$

Integrating once gives

$$\frac{1}{2} \sum_{i=1}^n m_i \dot{\mathbf{R}}_i^2 - \frac{1}{2}G \sum_{i=1}^n \sum_{j=1}^n \frac{m_i m_j}{r_{ij}} = E, \quad j \neq i. \quad (5.13)$$

where E is our tenth integration constant.

Note here that, similarly to Eq. (4.16),

$$\frac{d}{dt} \left(\frac{1}{2} \dot{\mathbf{R}}_i^2 \right) = \dot{\mathbf{R}}_i \cdot \ddot{\mathbf{R}}_i$$

and

$$\frac{d}{dt} \frac{1}{r_{ij}} = \frac{d}{dt} (\mathbf{r}_{ij} \cdot \mathbf{r}_{ij})^{-1/2} = -\frac{1}{2} (\mathbf{r}_{ij} \cdot \mathbf{r}_{ij})^{-3/2} 2\mathbf{r}_{ij} \cdot \dot{\mathbf{r}}_{ij} = -\frac{\dot{\mathbf{r}}_{ij} \cdot \mathbf{r}_{ij}}{r_{ij}^3}.$$

We define the quantity

$$U = \frac{1}{2}G \sum_{i=1}^n \sum_{j=1}^n \frac{m_i m_j}{r_{ij}}, \quad j \neq i,$$

so Eq. (5.13) becomes

$$T - U = E,$$

where

$$T = \frac{1}{2} \sum_{i=1}^n m_i \dot{\mathbf{R}}_i^2$$

is the kinetic energy of the system and $-U$ is its potential energy. The sum of T and U is conserved, but at any time there can be exchange of energy between potential and kinetic energy reservoirs.

5.4 The force function

The function

$$U = \frac{1}{2}G \sum_{i=1}^n \sum_{j=1}^n \frac{m_i m_j}{r_{ij}}, \quad j \neq i,$$

is a symmetric function of the masses and their mutual distances, but not of time or individual position vectors explicitly. Differentiating partially with respect to the position $\mathbf{R}_i$ we obtain

$$\nabla_i U = \begin{pmatrix} \partial U / \partial x_i \\ \partial U / \partial y_i \\ \partial U / \partial z_i \end{pmatrix}.$$

In the box below we show that

$$\nabla_i U = \mathbf{F}_i = G \sum_{j=1}^n \frac{m_i m_j}{r_{ij}^3} \mathbf{r}_{ij}, \quad i \neq j, \quad i = 1, 2, \dots, n.$$

Note here that

$$\nabla_i r_{ij}^{-1} = \nabla_i [(\mathbf{R}_j - \mathbf{R}_i)^2]^{-1/2} = -\frac{1}{2} [(\mathbf{R}_j - \mathbf{R}_i)^2]^{-3/2} 2(\mathbf{R}_j - \mathbf{R}_i) \nabla_i (\mathbf{R}_j - \mathbf{R}_i).$$

Since $\nabla_i \mathbf{R}_j = 0$ ($i \neq j$) and $\nabla_i \mathbf{R}_i = 1$ we have

$$\nabla_i r_{ij}^{-1} = [(\mathbf{R}_j - \mathbf{R}_i)^2]^{-3/2} (\mathbf{R}_j - \mathbf{R}_i) = \mathbf{r}_{ij} / r_{ij}^3,$$

and therefore

$$\nabla_i U = \mathbf{F}_i = G \sum_{j=1}^n \frac{m_i m_j}{r_{ij}^3} \mathbf{r}_{ij}, \quad i \neq j, \quad i = 1, 2, \dots, n.$$

The function U is called force function, because its partial derivatives give the components of the force vector. The work done by the gravitational force between two particles at $\mathbf{R}_1$ and $\mathbf{R}_2$ to move a particle from a position $\mathbf{R}_\infty$ to $\mathbf{R}_2$ is given by the line integral $\int_S \mathbf{F} \cdot d\mathbf{s}$, so

$$W_2 = \int_{\mathbf{R}_\infty}^{\mathbf{R}_2} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathbf{R}_\infty}^{\mathbf{R}_2} \left(\frac{\partial U}{\partial x_2} dx + \frac{\partial U}{\partial y_2} dy + \frac{\partial U}{\partial z_2} dz \right) = U(\mathbf{R}_2) - U(\mathbf{R}_\infty) = U(\mathbf{R}_2) = G \frac{m_1 m_2}{r_{12}}.$$

Moving now a third particle from infinity to position $\mathbf{R}_3$ by the forces between $\mathbf{R}_3$ and $\mathbf{R}_1$ and $\mathbf{R}_2$ gives the work

$$W_3 = G \left(\frac{m_1 m_3}{r_{13}} + \frac{m_2 m_3}{r_{23}} \right),$$

so that the total work to assemble three particles is

$$U = G \left(\frac{m_1 m_2}{r_{12}} + \frac{m_2 m_3}{r_{23}} + \frac{m_3 m_1}{r_{31}} \right),$$

which is equal to

$$U = \frac{1}{2}G \sum_{i=1}^3 \sum_{j=1}^3 \frac{m_i m_j}{r_{ij}},$$

and likewise for n particles

$$U = \frac{1}{2}G \sum_{i=1}^n \sum_{j=1}^n \frac{m_i m_j}{r_{ij}}.$$

5.5 The Virial Theorem

Let I be the moment of inertial of the system, defined by

$$I = \sum_{i=1}^n m_i \mathbf{R}_i^2.$$

Differentiating twice yields

$$\begin{aligned} \dot{I} &= \sum_{i=1}^n m_i 2 \mathbf{R}_i \dot{\mathbf{R}}_i. \\ \ddot{I} &= \sum_{i=1}^n m_i 2 (\dot{\mathbf{R}}_i^2 + \mathbf{R}_i \ddot{\mathbf{R}}_i), \end{aligned}$$

or

$$\ddot{I} = 4T + 2 \sum_{i=1}^n \mathbf{R}_i \nabla_i U, \quad (5.16)$$

In the box below we show that

$$\sum_{i=1}^n \mathbf{R}_i \nabla_i U = -U,$$

so that (5.16) becomes

$$\ddot{I} = 4T - 2U.$$

However, since $T - U = E$ we have

$$\ddot{I} = 4T + 2(E - T) = 2T + 2E.$$

If $E > 0$, and since $T \geq 0$, I must increase indefinitely and so at least one particle will escape. For the system to be bounded, $E < 0$ is a necessary (but not sufficient) condition.

Calculate

$$\sum_{i=1}^n \mathbf{R}_i \nabla_i U = \sum_{i=1}^n \sum_{j=1}^n \frac{m_i m_j}{r_{ij}^3} \mathbf{R}_i \cdot \mathbf{r}_{ij}.$$

The same expression can be generated by exchanging i and j , but since we differentiate then with respect to the other index the sign is different, so

$$\sum_{i=1}^n \mathbf{R}_i \nabla_i U = \sum_{i=1}^n \sum_{j=1}^n \frac{m_i m_j}{r_{ij}^3} (-\mathbf{R}_i) \cdot \mathbf{r}_{ij},$$

so that

$$\sum_{i=1}^n \mathbf{R}_i \nabla_i U = \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \frac{m_i m_j}{r_{ij}^3} (\mathbf{R}_i - \mathbf{R}_j) \cdot \mathbf{r}_{ij} = -\frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \frac{m_i m_j}{r_{ij}^3} \mathbf{r}_{ij} \cdot \mathbf{r}_{ij} = -U$$

If the system is statistically stationary then $\langle \ddot{I} \rangle = d^2 \langle \dot{I} \rangle / dt^2 = 0$, and so

$$2\langle T \rangle - \langle U \rangle = 0,$$

or

$$\langle T \rangle = \frac{1}{2} \langle U \rangle.$$

Thus, on the average, the kinetic energy is half the magnitude of the potential energy.

5.8 Lagrange's Solution of the Three-Body Problem

We now derive a very special solution of the three-body problem where the geometrical form of the three-body configuration is unchanged, except for an overall time-dependent scale factor. Such a solution exists only if the following three conditions are satisfied:

- $\mathbf{F}_i$ points to the centre of mass
- $F_i \propto R_i$ (force is proportional to distance)
- the initial $|\dot{\mathbf{R}}_i|$ is proportional to R_i and the directions of $\mathbf{R}_i \times \dot{\mathbf{R}}_i$ are the same for all $i = 1, 2, 3$.

We start with (5.1)–(5.3), applied to the case $n = 3$, ie

$$m_i \ddot{\mathbf{R}}_i = G \sum_{j=1}^3 \frac{m_i m_j}{r_{ij}^3} \mathbf{r}_{ij} \quad (j \neq i, \quad i = 1, 2, 3), \quad (5.17)$$

where

$$\mathbf{r}_{ij} = \mathbf{R}_j - \mathbf{R}_i, \quad \mathbf{r}_{ij} = -\mathbf{r}_{ji}.$$

We assume that the origin is the centre of mass, so

$$\sum_{i=1}^3 m_i \mathbf{R}_i = 0. \quad (5.18)$$

This can be rewritten in such a form that we extract the mutual distances, $\mathbf{r}_{12}$ and $\mathbf{r}_{13}$, in the form

$$M \mathbf{R}_1 = -m_2 \mathbf{r}_{12} - m_3 \mathbf{r}_{13}, \quad (5.19)$$

because that is just

$$(m_1 + m_2 + m_3) \mathbf{R}_1 = -m_2 (\mathbf{R}_2 - \mathbf{R}_1) - m_3 (\mathbf{R}_3 - \mathbf{R}_1),$$

which is satisfied because of Eq. (5.18). We shall return to (5.19) below, but for now we use it to show that the force on each particle, F_i , is proportional to the distance from the centre of mass, R_i . To this end we first take the square of (5.19),

$$M^2 \mathbf{R}_1^2 = m_2^2 r_{12}^2 + m_3^2 r_{13}^2 + 2m_2 m_3 \mathbf{r}_{12} \cdot \mathbf{r}_{13}. \quad (5.20)$$

If the shape of the configuration does not change, the relative distances between particles, r_{12} , r_{23} , and r_{31} are given by

$$\frac{r_{12}}{(r_{12})_0} = \frac{r_{23}}{(r_{23})_0} = \frac{r_{31}}{(r_{31})_0} = f(t),$$

where $(r_{ij})_0$ denotes the initial value of r_{ij} at $t = 0$. Also, if the angle between $\mathbf{r}_{12}$ and $\mathbf{r}_{13}$ is to be constant we must have

$$\dot{\theta}_1 = \dot{\theta}_2 = \dot{\theta}_3 = \dot{\theta}(t), \quad (5.22)$$

were $\dot{\theta}_i$ is the angular velocity of the particle i about the centre of mass. With these preparations (5.20) can be written in the form

$$M^2 \mathbf{R}_1^2 = [f(t)]^2 [m_2^2 (r_{12})_0^2 + m_3^2 (r_{13})_0^2 + 2m_2 m_3 (r_{12})_0 (r_{13})_0 \cos \alpha_1], \quad (5.24)$$

where α_1 is the (constant) angle between $\mathbf{r}_{12}$ and $\mathbf{r}_{13}$. Equation (5.24) shows that

$$R_1 = (R_1)_0 f(t),$$

or, in general,

$$R_i = (R_i)_0 f(t), \quad (5.25)$$

We now use polar coordinates to write down the angular momentum integral,

$$\mathbf{C} = \sum_{i=1}^3 m_i \mathbf{R}_i \times \dot{\mathbf{R}}_i, \quad (5.23)$$

where

$$\mathbf{R}_i = \hat{\mathbf{I}}_i R_i \quad \text{and} \quad \dot{\mathbf{R}}_i = \hat{\mathbf{I}}_i \dot{R}_i + \hat{\mathbf{J}}_i R_i \dot{\theta}_i.$$

Thus,

$$\mathbf{R}_i \times \dot{\mathbf{R}}_i = \hat{\mathbf{K}}_i R_i^2 \dot{\theta}_i.$$

However, all three particles are in one and the same plane, the normal of which is given by $\hat{\mathbf{K}}_1 = \hat{\mathbf{K}}_2 = \hat{\mathbf{K}}_3 \equiv \hat{\mathbf{K}}$, where $\hat{\mathbf{K}}_i = \hat{\mathbf{I}}_i \times \hat{\mathbf{J}}_i$. Therefore, and because of (5.22) and (5.25), we have

$$C = [f(t)]^2 \dot{\theta} \sum_{i=1}^3 [m_i (R_i)_0^2]. \quad (5.26)$$

The angular momentum of each particle about the centre of mass is $\mathbf{R}_i^2 \dot{\theta} = (f^2 \dot{\theta}) [m_i (R_i)_0^2]$. Now, $[m_i (R_i)_0^2]$ is obviously time-independent, but the angular momentum of each particle could still change if $[f(t)]^2 \dot{\theta}$ changes. However, Eq. (5.26) shows that because $\sum_{i=1}^3 [m_i (R_i)_0^2]$ is constant in time, also $[f(t)]^2 \dot{\theta}$ must be constant. Thus, *the angular momentum of each particle about the centre of mass is constant*. There is still something else we can learn from (5.26). In polar coordinates, the force (per unit mass) acting on each particle, ie the acceleration, is

$$\mathbf{F}_i = \ddot{\mathbf{R}}_i = \hat{\mathbf{I}}_i (\ddot{R}_i - R_i \dot{\theta}^2) + \hat{\mathbf{J}}_i \left[\frac{1}{R_i} \frac{d}{dt} (R_i^2 \dot{\theta}) \right].$$

(This is analogous to the corresponding expression for $\ddot{\mathbf{r}}$ in the two-body problem.) Now, because the angular momentum of each particle about the centre of mass is constant, the term multiplying $\hat{\mathbf{J}}_i$ vanishes and so *the force acting on each particle is directed to the centre of mass*.

As we have just seen, the force per unit mass $\mathbf{F}_i$, is directed along the radius vector $\mathbf{R}_i$, and its magnitude satisfies

$$F_i = \ddot{R}_i - R_i \dot{\theta}^2.$$

Using (5.25) we see that

$$F_i = \ddot{f} (R_i)_0 - R_i \dot{\theta}^2 = R_i (\ddot{f}/f - \dot{\theta}^2).$$

Since the expression $(\ddot{f}/f - \dot{\theta}^2)$ is just a function of time and independent of the particle, we see that the ratio between any two forces is equal to the ratio of the distances from the centre of mass, ie

$$F_1 : F_2 : F_3 = R_1 : R_2 : R_3 \quad (5.27)$$

We now consider two cases, which satisfy the three conditions given above. Since the force is directed along the radius vector we have

$$\mathbf{R}_i \times \mathbf{F}_i = 0,$$

or

$$\mathbf{R}_i \times \ddot{\mathbf{R}}_i = 0.$$

The acceleration of particle 1 is given by

$$\ddot{\mathbf{R}}_1 = G \left[\frac{m_2}{r_{12}^3} (\mathbf{R}_2 - \mathbf{R}_1) + \frac{m_3}{r_{13}^3} (\mathbf{R}_3 - \mathbf{R}_1) \right],$$

see (5.17). We now take the dot product of $\ddot{\mathbf{R}}_1$ with $\mathbf{R}_1$ and obtain

$$\mathbf{R}_1 \times \left(m_2 \frac{\mathbf{R}_2}{r_{12}^3} + m_3 \frac{\mathbf{R}_3}{r_{13}^3} \right) = 0. \quad (5.28)$$

We now apply (5.18) to substitute for

$$m_3 \mathbf{R}_3 = -m_1 \mathbf{R}_1 - m_2 \mathbf{R}_2$$

in (5.28), which then becomes

$$\mathbf{R}_1 \times \left(m_2 \frac{\mathbf{R}_2}{r_{12}^3} - m_2 \frac{\mathbf{R}_2}{r_{13}^3} \right) = 0.$$

or

$$(\mathbf{R}_1 \times \mathbf{R}_2) \left(\frac{1}{r_{12}^3} - \frac{1}{r_{13}^3} \right) = 0.$$

There are of course two similar equations for the other two particles that must also hold, ie

$$(\mathbf{R}_2 \times \mathbf{R}_3) \left(\frac{1}{r_{23}^3} - \frac{1}{r_{21}^3} \right) = 0.$$

and

$$(\mathbf{R}_3 \times \mathbf{R}_1) \left(\frac{1}{r_{31}^3} - \frac{1}{r_{32}^3} \right) = 0.$$

This set of equations can be satisfied in two different ways: *either* by having

$$r_{12} = r_{23} = r_{31} \equiv r,$$

which corresponds to an equilateral triangle, or by having

$$\mathbf{R}_1 \times \mathbf{R}_2 = \mathbf{R}_2 \times \mathbf{R}_3 = \mathbf{R}_3 \times \mathbf{R}_1 = 0,$$

which puts all the particles in a straight line.

In the first case of an equilateral triangle Eq. (5.17) becomes

$$m_1 \ddot{\mathbf{R}}_1 = \frac{Gm_1}{r^3} (m_2 \mathbf{r}_{12} + m_3 \mathbf{r}_{13}).$$

Here is where our boxed equation (5.19) comes in handy, because the stuff on the rhs, $(m_2 \mathbf{r}_{12} + m_3 \mathbf{r}_{13})$, is just equal to $M \mathbf{R}_1$, so that

$$\ddot{\mathbf{R}}_1 = -\frac{GM}{r^3} \mathbf{R}_1. \quad (5.29)$$

This can be written in slightly different form by expressing r in terms of R_1 . This can be done using Eq. (5.20) which, using the fact that the angle between $\mathbf{r}_{12}$ and $\mathbf{r}_{13}$ is 60° , and $\cos 60^\circ = 1/2$, takes the form

$$M^2 \mathbf{R}_1^2 = (m_2^2 + m_3^2 + m_2 m_3) r^2. \quad (5.29b)$$

Substituting this in (5.29) yields

$$\ddot{\mathbf{R}}_1 = -\frac{GM_1}{R_1^3} \mathbf{R}_1, \quad (5.30)$$

where

$$M_1 = \frac{(m_2^2 + m_3^2 + m_2 m_3)^{3/2}}{(m_1 + m_2 + m_3)^2}. \quad (5.31)$$

This equation is just like the equation for the two-body problem; see Eq. (4.19), but the effective mass is not m_1 , but M_1 . As long as the initial conditions fall in the present class of solutions, the arrangement will stay an equilateral triangle, though its size may oscillate or grow/shrink.

In the second case of a collinear solution the force (per mass) acting on particle 1 is

$$F_1 = Gm_2 \frac{x_2 - x_1}{x_{12}} + Gm_3 \frac{x_3 - x_1}{x_{13}}.$$

But by Eq. (5.25)

$$x_i = (x_i)_0 f(t),$$

so that

$$F_1 = \frac{G}{f^2} \left[m_2 \frac{x_2 - x_1}{x_{12}} + m_3 \frac{x_3 - x_1}{x_{13}} \right]_0 = \frac{\text{constant}}{f^2}.$$

This equation shows that, since f is proportional to the distance, the force is inversely proportional to the square of the separation. We now adopt the additional constraint that the force is directly proportional to the distance from the centre of mass, ie

$$F_1 : F_2 : F_3 = x_1 : x_2 : x_3,$$

which fixes then all distances. Thus, we require $F_i = Ax_i$ and write

$$Ax_1 = m_2 \frac{x_2 - x_1}{x_{12}^3} + m_3 \frac{x_3 - x_1}{x_{13}^3}, \quad (5.32)$$

$$Ax_2 = m_3 \frac{x_3 - x_2}{x_{23}^3} + m_1 \frac{x_1 - x_2}{x_{21}^3}, \quad (5.33)$$

$$Ax_3 = m_1 \frac{x_1 - x_3}{x_{31}^3} + m_2 \frac{x_2 - x_3}{x_{32}^3}. \quad (5.34)$$

These three particles are arranged in the order 3, 2, 1, so $x_1 - x_2 > 0$ and $x_2 - x_3 > 0$.

Therefore the last three equations become

$$Ax_1 = -\frac{m_2}{x_{12}^2} - \frac{m_3}{x_{13}^2}, \quad (5.32')$$

$$Ax_2 = -\frac{m_3}{x_{23}^2} + \frac{m_1}{x_{21}^2}, \quad (5.33')$$

$$Ax_3 = +\frac{m_1}{x_{31}^2} + \frac{m_2}{x_{32}^2}. \quad (5.34')$$

Subtract (5.33') from (5.32'), i.e. calculate (5.32') - (5.33'), which gives

$$Ax_{12} = m_3 \left(\frac{1}{x_{23}^2} - \frac{1}{x_{13}^2} \right) - (m_1 + m_2) \frac{1}{x_{21}^2}, \quad (5.35)$$

while subtracting (5.34') from (5.33') gives

$$Ax_{23} = m_1 \left(\frac{1}{x_{12}^2} - \frac{1}{x_{13}^2} \right) - (m_2 + m_3) \frac{1}{x_{23}^2}, \quad (5.36)$$

We now define the ratio

$$X = \frac{x_{23}}{x_{12}},$$

and, because of the way the particles are arranged, we have $x_{13} = x_{12} + x_{23}$, so

$$\frac{x_{13}}{x_{12}} = X + 1, \quad \frac{x_{12}}{x_{13}} = \frac{1}{X + 1}, \quad \frac{x_{23}}{x_{13}} = \frac{X}{X + 1}.$$

Multiplying (5.35) by x_{23}^2 yields

$$Ax_{12}x_{23}^2 = m_3 \left[1 - \frac{X^2}{(X + 1)^2} \right] - (m_1 + m_2) X^2, \quad (5.35')$$

whilst multiplying (5.36) by $x_{12}x_{23}$ yields

$$Ax_{12}x_{23}^2 = m_1 \left[X - \frac{X}{(X + 1)^2} \right] - (m_2 + m_3) \frac{1}{X}, \quad (5.36')$$

Subtracting (5.36') from (5.35') gives

$$m_3 \left[1 - \frac{X^2}{(X + 1)^2} \right] - (m_1 + m_2) X^2 - m_1 \left[X - \frac{X}{(X + 1)^2} \right] + (m_2 + m_3) \frac{1}{X} = 0.$$

Multiplying by $X(X + 1)^2 = X^3 + 2X^2 + X$ gives

$$m_3 [X^3 + 2X^2 + X - X^3] - (m_1 + m_2) X^3(X + 1)^2 - m_1 [X^2(X + 1)^2 - X^2] + (m_2 + m_3) (X + 1)^2 = 0,$$

or

$$m_3 [2X^2 + X] - (m_1 + m_2) X^5 + 2X^4 + X^3 - m_1 [X^4 + 2X^3] + (m_2 + m_3) X^2 + 2X + 1 = 0.$$

Collecting similar powers of X gives

$$\begin{aligned} X^5(m_1 + m_2) + X^4(2m_1 + 2m_2 + m_1) + X^3(m_1 + m_2 + 2m_1) + X^2(-2m_3 - m_2 - m_3) \\ + X(-m_3 - 2m_2 - 2m_3) + (-m_2 - m_3) = 0, \end{aligned}$$

or

$$X^5(m_1 + m_2) + X^4(3m_1 + 2m_2) + X^3(3m_1 + m_2) - X^2(m_2 + 3m_3) - X(2m_2 + 3m_3) - (m_2 + m_3) = 0.$$

Descartes sign rule states that the number of positive roots of a polynomial is equal to the number of sign changes in the sequence of the coefficients of the polynomial. Note that there is only one change of sign (between the coefficients of X^3 and X^2). Hence, there can only be one positive root! This then defines uniquely the positions of the three particles in the order chosen. In the table below some results for X are given for different values of m_1 and m_2 , keeping $m_3 = 1$ in all cases.

	$m_1 = 0$	0.01	0.1	0.3	1
$m_2 = 0$	—	6.07	2.54	1.64	1
0.01	6.40	4.86	2.47	1.63	1
0.1	2.88	2.76	2.10	1.54	1
0.3	2.00	1.97	1.73	1.42	1
1	1.43	1.42	1.36	1.24	1

5.9 Remarks on Lagrange's Solutions

Both the equilateral triangle and the straight line solutions are realised in the solar system. Once two particles are taken as points of reference, there are 5 points where the third particle can be placed. Those points are called L_1 , L_2 , L_3 , L_4 , and L_5 . The points L_4 and L_5 form the equilateral triangle. In the solar system there are 12 asteroids in the places formed by the Jupiter–Sun system. Those points are also called the *Trojans*. Their positions can oscillate about the actual L_4 and L_5 points with small amplitude. Finally, the straight line solution appears to be realised in the form of additional dust on the direction earth–sun away from the sun.

5.10 The Circular Restricted Three-Body Problem

In this problem the third mass is negligible, so there are only two masses. We simplify notation by assuming that the sum of the two masses is unity. This corresponds to saying that we measure mass in units of the total mass. Apart from that everything else is the same as in Section 5.8. In particular, we will be dealing with either the collinear or the triangular solution of the 3-body problem.

5.10.1 Jacobi's integral

In the following we denote the positions of the three bodies by $\mathbf{r}_0$ (for the massless body), $\mathbf{r}_1$ (for the more massive body) and $\mathbf{r}_2$ (for the less massive body). The orbits of m_1 and m_2 about the common centre of mass are easy to write down:

$$\begin{aligned} x_1 &= R_1 \cos \omega t, & y_1 &= R_1 \sin \omega t, \\ x_2 &= -R_2 \cos \omega t, & y_2 &= -R_2 \sin \omega t, \end{aligned}$$

where $\omega = \dot{\theta} = \text{const}$ is the angular velocity of the two bodies. The position of the third body is then either on a straight line that connects the two bodies, or in one of the two triangular points.

The motion of the third (massless) particle is governed by the equation

$$m_0 \ddot{\mathbf{r}}_0 = G \frac{m_0 m_1}{r_{01}^3} \mathbf{r}_{01} + G \frac{m_0 m_2}{r_{02}^3} \mathbf{r}_{02}.$$

Although $m_0 \rightarrow 0$, this equation cannot be neglected, because the factor m_0 cancels. The only effect of the neglect of m_0 is in the equation of the two other bodies, m_1 and m_2 , which constitute now a two-body problem in its simplest form.

The equations can be nondimensionalised by measuring $\mathbf{r}$ in units of the separation between the two bodies, $a = x_2 - x_1$, mass in units of the total mass $M = m_1 + m_2$, and time in units of $\sqrt{a^3/GM}$. In such units the massless body is governed by

$$\ddot{\mathbf{r}}_0 = (1 - \mu) \frac{\mathbf{r}_{01}}{r_{01}^3} + \mu \frac{\mathbf{r}_{02}}{r_{02}^3}.$$

In component form, $\mathbf{r} = (\xi, \eta, \zeta)$, we have

$$\begin{aligned} \ddot{\xi} &= (1 - \mu) \frac{\xi_1 - \xi}{r_1^3} + \mu \frac{\xi_2 - \xi}{r_2^3}, \\ \ddot{\eta} &= (1 - \mu) \frac{\eta_1 - \eta}{r_1^3} + \mu \frac{\eta_2 - \eta}{r_2^3}, \\ \ddot{\zeta} &= (1 - \mu) \frac{\zeta_1 - \zeta}{r_1^3} + \mu \frac{\zeta_2 - \zeta}{r_2^3}, \end{aligned} \tag{5.38}$$

where

$$\begin{aligned} r_1^2 &= (\xi_1 - \xi)^2 + (\eta_1 - \eta)^2 + (\zeta_1 - \zeta)^2, \\ r_2^2 &= (\xi_2 - \xi)^2 + (\eta_2 - \eta)^2 + (\zeta_2 - \zeta)^2. \end{aligned}$$

The positions of m_1 and m_2 are

$$x_1 = -\mu, \quad x_2 = 1 - \mu.$$

Within a *rotating* frame of reference, (x, y, z) , we have

$$r_1^2 = (x_1 - x)^2 + y^2 + z^2,$$

$$r_2^2 = (x_2 - x)^2 + y^2 + z^2,$$

where the line $y = z = 0$ marks the connection line between the two masses. The rotating frame rotates with the angular velocity $\dot{\theta} = \sqrt{G(m_1 + m_2)/a^3} = 1$, because $a = x_2 - x_1 = 1$ and $G(m_1 + m_2) = 1$.

Dimensionless units: a proper change of variables. In dimensional units the total mass is $M = m_1 + m_2$ and the semimajor axis is $a = x_2 - x_1$. We then define new dimensionless quantities as follows:

$$\tilde{x} = x/a, \quad \tilde{m} = m/M.$$

The orbital period is $2\pi\sqrt{GM/a^3}$, and so we could introduce a dimensionless time by dividing by the orbital period. Instead, in order to avoid 2π factors all over the place, we define the dimensionless time to be

$$\tilde{t} = t/\sqrt{GM/a^3}.$$

In the following we drop the tildes. The angular velocity is then unity, which is quite handy.

We now adopt the coordinate transformation

$$\left. \begin{aligned} \xi &= x \cos t - y \sin t \\ \eta &= x \sin t + y \cos t \\ \zeta &= z \end{aligned} \right\}, \quad (5.39)$$

We need the second time derivatives of ξ and η , so we differentiate twice,

$$\dot{\xi} = \dot{x} \cos t - \dot{y} \sin t - x \sin t - y \cos t$$

$$\ddot{\xi} = \ddot{x} \cos t - \ddot{y} \sin t - 2\dot{x} \sin t - 2\dot{y} \cos t - x \cos t + y \sin t,$$

or just

$$\ddot{\xi} = (\ddot{x} \cos t - 2\dot{y} - x) \cos t - (\ddot{y} + 2\dot{x} - y) \sin t.$$

For the η derivatives we have

$$\ddot{\eta} = (\ddot{x} \cos t - 2\dot{y} - x) \sin t + (\ddot{y} + 2\dot{x} - y) \cos t.$$

We insert this into Eq. (5.38), order by $\cos t$ and $\sin t$, and note that the resulting equation can only be satisfied when the coefficients in front of $\cos t$ and $\sin t$ vanish. Thus, this coordinate transformation leads to the following system of equations

$$\left. \begin{aligned} \ddot{x} - 2\dot{y} - x &= -(1 - \mu) \frac{x - x_1}{r_1^3} - \mu \frac{x - x_2}{r_2^3} \\ \ddot{y} + 2\dot{x} - y &= - \left(\frac{1 - \mu}{r_1^3} + \frac{\mu}{r_2^3} \right) y \\ \ddot{z} &= - \left(\frac{1 - \mu}{r_1^3} + \frac{\mu}{r_2^3} \right) z \end{aligned} \right\}, \quad (5.41)$$

which does no longer involve t explicitly!

We can now define a function U (an effective negative potential) by

$$U = \frac{1}{2}(x^2 + y^2) + \frac{1-\mu}{r_1^3} + \frac{\mu}{r_2^3},$$

which allows us to write (5.41) in the form

$$\ddot{x} - 2\dot{y} = \frac{\partial U}{\partial x}, \quad (5.42)$$

$$\ddot{y} + 2\dot{x} = \frac{\partial U}{\partial y}, \quad (5.43)$$

$$\ddot{z} = \frac{\partial U}{\partial z}. \quad (5.44)$$

Note that we are in a rotating frame of reference. Therefore $-U$ is now the usual potential energy supplemented by a potential for the so-called Coriolis force, $\frac{1}{2}(x^2 + y^2)$. Plots of the effective potential $-U$ are given in Figs. 5.2 and 5.3.

The five Lagrange points must be where $\nabla U = 0$, because then $\dot{x} = \dot{y} = \dot{z} = 0$. Those points are also called libration points, which is maybe best translated as *equilibrium points*. For the collinear solution we have $y = z = 0$ and so

$$U = \frac{1}{2}x^2 + \frac{1-\mu}{|x_1 - x|} + \frac{\mu}{|x_2 - x|}.$$

A plot of the effective potential, $-U$, is given in Fig. 5.4.

5.10.2 Lagrange points revisited

For the L_1 point we have the ordering $x > x_1$ and $x_2 > x$. Thus, we can resolve the modulus sign by writing:

$$U = \frac{1}{2}x^2 + \frac{1-\mu}{x - x_1} + \frac{\mu}{x_2 - x}.$$

At the L_1 point we have $\partial U / \partial x = 0$, where

$$\frac{\partial U}{\partial x} = x - \frac{1-\mu}{(x - x_1)^2} + \frac{\mu}{(x_2 - x)^2}.$$

We now want to introduce the distance ratio $X = (x - x_1)/(x_2 - x)$. Because

$$X + 1 = \frac{(x - x_1) - (x_2 - x)}{x_2 - x} = \frac{x_2 - x_1}{x_2 - x} = \frac{1}{x_2 - x}$$

where can substitute

$$x_2 - x = \frac{1}{X + 1}, \quad \text{and} \quad x - x_1 = \frac{X}{X + 1}.$$

Thus, the last two terms in the expression for U are readily expressed in terms of X . However, for the first term, x , we remember that the centre of mass is located at the position

$$x_c = (1 - \mu)x_1 + \mu x_2 = 0,$$

because $x_1 = -\mu$ and $x_2 = 1 - \mu$. Thus

$$x = x - x_c = [(1 - \mu) + \mu]x - (1 - \mu)x_1 - \mu x_2 = (1 - \mu)(x - x_1) - \mu(x_2 - x),$$

or

$$x = (1 - \mu) \frac{X}{X + 1} - \mu \frac{1}{X + 1}$$

With these preparations we write $\partial U/\partial x$ as

$$\frac{\partial U}{\partial x} = 0 = (1 - \mu) \frac{X}{X+1} - \mu \frac{1}{X+1} - (1 - \mu) \frac{(X+1)^2}{X^2} + \mu(X+1)^2.$$

Multiplying by $(X-1)X^2$ leads to

$$0 = (1 - \mu)X^3 - \mu X^2 - (1 - \mu)(X+1)^3 + \mu(X+1)^3 X^2.$$

Which is a quintic equation:

$$\mu(X^5 + 3X^4 + 3X^3 + X^2 - X^2) - (1 - \mu)(X^3 + 3X^2 + 3X + 1 - X^3) = 0,$$

or

$$\mu(X^5 + 3X^4 + 3X^3) - (1 - \mu)(3X^2 + 3X + 1) = 0.$$

This is just the same equation as (5.37) using $m_1 = \mu$, $m_2 = 0$, and $m_3 = 1 - \mu$. A quintic equation can be solved by iteration. In this case it may be simpler, however, to calculate instead μ for a number of different values of X ; see the table below.

X	μ	x_1	x_0	x_2	r_1	r_2
1	1/2	-1/2	1/2	1/2	1/2	1/2
2	1.545×10^{-1}	-0.1545	0.5122	0.8455	0.6667	0.3333
5	1.665×10^{-2}	-0.0166	0.8167	0.9834	0.8333	0.1667
10	2.483×10^{-3}	-0.0025	0.9066	0.9975	0.9091	0.0909
100	2.940×10^{-6}	0.0000	0.9901	1.0000	0.9901	0.0099
1000	2.994×10^{-9}	0.0000	0.9990	1.0000	0.9990	0.0010

5.10 Stability of the libration points

The evolution of points in the three-body problem can easily be calculated numerically. In the lecture we have seen the evolution of a system with two masses of fractional mass $\mu = 0.1$ and $\mu = 0.02$ and a third body of zero mass at one of the five libration points. For $\mu = 0.1$ the massless point at the L_1 point stayed in place for about half an orbit and then ran away. For the L_4 or L_5 points it stayed in place much longer (about 5 orbits), before it finally ran away. However, for $\mu = 0.02$ a point at L_4 or L_5 carried on for at least 20 orbits.

Whether a point stays in place or not is determined by the stability of the solution. To calculate the stability it suffices to consider only *small departures* from the solution. We denote a particular solution by (x_0, y_0, z_0) and consider a slightly displaced point

$$(x_0 + \xi, y_0 + \eta, z_0 + \zeta).$$

The evolution of the perturbation vector (ξ, η, ζ) is well described by the linearised equations that are obtained by Taylor expansion of the rhs of (5.42)–(5.44) about the solution (x_0, y_0, z_0) . The rhs of (5.42)–(5.44) is the gradient of U , so the first nontrivial term in the Taylor expansion starts with second derivatives.

$$\left. \begin{aligned} \ddot{\xi} - 2\dot{\eta} &= \xi U_{xx} + \eta U_{xy} + \zeta U_{xz}, \\ \dot{\eta} + 2\dot{\xi} &= \xi U_{yx} + \eta U_{yy} + \zeta U_{yz}, \\ \ddot{\zeta} &= \xi U_{zx} + \eta U_{zy} + \zeta U_{zz}, \end{aligned} \right\} \quad (5.56)$$

where

$$U_{xx} = \left(\frac{\partial^2 U}{\partial x^2} \right)_0, \quad U_{xy} = \left(\frac{\partial^2 U}{\partial x \partial y} \right)_0 = U_{yx}, \quad \text{etc.}$$

Note that these are just *constant coefficients*. Hence, eq. (5.56) is a system of ordinary differential equations with constant coefficients, which is easy to solve. For simplicity we consider now the two-dimensional case in the xy plane, ignoring thus the z direction. Then

$$\left. \begin{aligned} \ddot{\xi} - 2\dot{\eta} &= \xi U_{xx} + \eta U_{xy}, \\ \dot{\eta} + 2\dot{\xi} &= \xi U_{yx} + \eta U_{yy}. \end{aligned} \right\} \quad (5.57)$$

We assume the solution to be of the form

$$\xi = \xi_0 e^{\lambda t}, \quad \eta = \eta_0 e^{\lambda t},$$

which turns the equations into a system of algebraic equations

$$\left. \begin{aligned} \lambda^2 \xi_0 - 2\lambda \eta_0 &= \xi_0 U_{xx} + \eta_0 U_{xy}, \\ \lambda^2 \eta_0 + 2\lambda \xi_0 &= \xi_0 U_{xy} + \eta_0 U_{yy}. \end{aligned} \right\}$$

or

$$\left. \begin{aligned} (\lambda^2 - U_{xx})\xi_0 - (2\lambda + U_{xy})\eta_0 &= 0, \\ (\lambda^2 - U_{yy})\eta_0 - (-2\lambda + U_{xy})\xi_0 &= 0. \end{aligned} \right\}$$

We can now eliminate η_0 , for example. Thus, we multiply the first equation by $(\lambda^2 - U_{yy})$, and substitute for $(\lambda^2 - U_{yy})\eta_0$ the term $(-2\lambda + U_{xy})\xi_0$. This gives

$$[(\lambda^2 - U_{xx})(\lambda^2 - U_{yy}) - (2\lambda + U_{xy})(-2\lambda + U_{xy})]\xi_0 = 0.$$

one possibility to satisfy this equation is that $\xi_0 = 0$, but this “trivial” solution is of no interest. The other possibility is that

$$(\lambda^2 - U_{xx})(\lambda^2 - U_{yy}) - (2\lambda + U_{xy})(-2\lambda + U_{xy}) = 0$$

This is a quadratic equation for λ^2 ,

$$(\lambda^2)^2 - \lambda^2(U_{xx} + U_{yy} - 4) + U_{xx}U_{yy} - U_{xy}^2 = 0. \quad (5.59)$$

Calculation of the coefficients

In order to calculate the coefficients U_{xx} , U_{xy} , and U_{yy} we need to differentiate U twice. Recall

$$U = \frac{1}{2}(x^2 + y^2) + \frac{1-\mu}{r_1^3} + \frac{\mu}{r_2^3},$$

where

$$r_i^2 = (x - x_i)^2 + y^2,$$

assuming always $z = 0$. The x -positions of the two bodies is

$$x_1 = -\mu, \quad x = 1 - \mu.$$

We define

$$A = \frac{1-\mu}{r_1^3} + \frac{\mu}{r_2^3},$$

$$B = 3 \left(\frac{1-\mu}{r_1^5} + \frac{\mu}{r_2^5} \right),$$

$$C = 3 \left[\frac{1-\mu}{r_1^5}(x_0 - x_1) + \frac{\mu}{r_2^5}(x_0 - x_2) \right],$$

and find that

$$U_{xx} = 1 - A + 3 \left[\frac{1-\mu}{r_1^5}(x_0 - x_1)^2 + \frac{\mu}{r_2^5}(x_0 - x_2)^2 \right],$$

$$U_{yy} = 1 - A + B y_0^2,$$

$$U_{xy} = C y_0.$$

We can now apply this both to the straight line solutions as well as to the triangular solutions. We start with the latter.

Stability of the L_4 and L_5 points

For the L_4 and L_5 points we have

$$x_0 = \frac{1}{2}(-\mu + 1 - \mu) = \frac{1}{2} - \mu,$$

$$y_0 = \pm \sqrt{1 - \frac{1}{2^2}} = \pm \sqrt{\frac{3}{4}} = \pm \frac{1}{2}\sqrt{3}$$

and, of course,

$$x_1 = -\mu, \quad x_2 = 1 - \mu,$$

so we have

$$x_0 - x_1 = \frac{1}{2}, \quad x_0 - x_2 = -\frac{1}{2}.$$

Finally we recall that for the triangular solution we have

$$r_1 = r_2 = 1.$$

With this we can calculate

$$A = 1 - \mu + \mu = 1$$

$$B = 3(1 - \mu + \mu) = 3$$

$$C = 3[\frac{1}{2}(1 - \mu) - \frac{1}{2}\mu] = 3(\frac{1}{2} - \mu)$$

Thus, we have

$$U_{xx} = 3/4, \quad U_{yy} = 9/4, \quad U_{xy} = \pm \frac{3}{4}\sqrt{3}(1 - 2\mu).$$

which leads to the equation

$$\lambda^4 + \lambda^2 + \frac{27}{4}\mu(1 - \mu) = 0,$$

with the solution

$$(\lambda^2)_{1,2} = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{27}{4}\mu(1 - \mu)}$$

or

$$(\lambda^2)_{1,2} = \frac{1}{2} \left[1 \pm \sqrt{1 - 27\mu(1 - \mu)} \right].$$

In order that λ^2 is real, we have to require that $1 - 27\mu(1 - \mu) \geq 0$. Now, since $\mu < 1/2$, this means that $\mu \leq (1/2) - \sqrt{(23/108)} = 0.0385$ (ie only the minus sign applies). Thus, for $\mu \leq 0.0385$ the characteristic values λ are negative, so the Trojan points are stable. Otherwise they are not.

Example: for $\mu = 1/11 = 0.09$ we have $(\lambda^2)_{1,2} = -0.5 \pm 0.55i$, which gives $\lambda_{1,2\pm} = \pm(0.35 \pm 0.79i)$, so, explicitly, the four solutions are

$$\lambda_{1+} = 0.35 + 0.79i, \quad \lambda_{2+} = 0.35 - 0.79i, \quad \lambda_{1-} = -0.35 - 0.79i, \quad \lambda_{2-} = -0.35 + 0.79i.$$

The important thing here is that there is *at least one* solution whose real part is *positive*. In this case $\max(\text{Re}\lambda) = 0.35$. This means the perturbation will grow exponentially in time with an e -folding time of $1/0.35 \approx 3$.

Another example: $\mu = 1/101$ gives

$$\lambda_1^2 = -0.07, \quad \lambda_2^2 = -0.93,$$

so

$$\lambda_{1\pm} = \pm 0.27i, \quad \lambda_{2\pm} = \pm 0.96i.$$

In this case there is no real part, so all four solutions are stable (or at least not unstable).

Stability of the L_1 point

For the points on the straight line solution we have

$$U_{xx} = 1 + 2A, \quad U_{yy} = 1 - A, \quad U_{xy} = 0,$$

where

$$A = \frac{1 - \mu}{r_1^3} + \frac{\mu}{r_2^3}.$$

The characteristic equation is then

$$\lambda^4 + (2 - A)\lambda^2 + (1 + A - 2A^2) = 0. \quad (5.1)$$

The last term in this equation is always negative, ie $1 + A - 2A^2 < 0$ (see figure 5.5), so there is always a positive real part for one of the four solutions. This means instability.

Example: for $\mu = 1/11 = 0.09$ we have $(\lambda^2)_1 = 11.3$ and $(\lambda^2)_2 = -6.8$, which gives $\lambda_{1\pm} = \pm 3.4$ and $\lambda_{2\pm} = \pm 2.6i$, so, explicitly, the four solutions are

$$\lambda_{1+} = 3.4, \quad \lambda_{2+} = 2.6i, \quad \lambda_{1-} = -3.4, \quad \lambda_{2-} = -2.6i.$$

Again, the important thing here is that there is *at least one* solution whose real part is *positive*. In this case $\max(\text{Re}\lambda) = 3.4$. This means the perturbation will grow exponentially in time with an *e*-folding time of $1/3.4 \approx 0.3$. In other words, the instability will show up approximately 10 times earlier than for the L_4 and L_5 points!

Figure 5.1: Location of the three particles.

Figure 5.2: Level contours of the effective potential, $-U$, for the case with $\mu = 0.1$ in the $x - y$ plane.

Figure 5.3: Surface plot of the effective potential, $-U$, for the case with $\mu = 0.1$ in the $x - y$ plane.

Figure 5.4: Effective potential, $-U$, for two cases with $\mu = 0.1$ and $\mu = 0.01$.

Figure 5.5: For given values of X (remember: $X > 1$ for $\mu < \frac{1}{2}$) we can calculate a bunch of values of mu (first panel). For those values we calculate then first r_1 (solid line) and r_2 (dotted line), and then A , and finally $1 + A - 2A^2$. We see that always $2 - A < 0$ and $1 + A - 2A^2 < 0$.

Chapter 11

Rocket Dynamics and Transfer Orbits

With the development of interplanetary space travel the theory of celestial mechanics has received increased attention. No longer do we only want to understand existing orbits, we also want to manipulate orbits, get around and go to other orbits. This is accomplished by firing rocket motors. That's why we first discuss the theory of rockets.

11.2 Motion of a Rocket

Fig. 11.1 illustrates the basic principle. The total momentum is conserved, so the sum of the changes of momentum of the cannon ball, $v dm$, and the car, $m dv$, has to vanish, i.e.

$$m dv + v dm = 0. \quad (11.1)$$

This is really another way of saying that the total change of momentum vanishes, i.e. $d(mv) = 0$, because $d(mv) = m dv + v dm$ by the use of the chain rule. Here, m is the mass of the rockets, but this mass is of course constantly decreasing if the rocket motor is on. In each time unit the mass of the rocket decreases by an amount dm . In a rocket the exhaust velocity v_e is constant, we have

$$m dv + v_e dm = 0.$$

or

$$dv = -v_e \frac{dm}{m},$$

so integration leads to

$$\int_{v_0}^v dv' = -v_e \int_{m_0}^m \frac{dm'}{m'},$$

where v' and m' denote integration variables. Thus we have

$$v - v_0 = v_e \ln(m_0/m). \quad (11.2)$$

11.2.1 Motion of a Rocket in a Gravitational Field

Now the gravitational force leads to an acceleration, so the impulse is no longer conserved and we have

$$\frac{d}{dt}(mv) = mg,$$

which leads to

$$dv = -v_e \frac{dm}{m} + g dt,$$

which can be integrated to give

$$v - v_0 = v_e \ln(m_0/m) + \int_0^t g(t') dt'. \quad (11.2)$$

Here g is a function of time because g depends on height. In the case of the earth, $g = GM/r^2$ where $r = R_E + h$ with R_E being the radius of the earth. Then,

$$h(t) = \int_0^t v(t') dt',$$

which needs to be solved simultaneously.

11.3 Transfer Between Orbits in a Single Central Force

11.3.1 Transfer between circular, coplanar orbits

We use the change in the energy constant to calculate the energy necessary to transfer from one to another orbit. The energy constant for a circular orbit is $C = -\mu/(2a)$. Applied to the transfer from an orbit with radius a_1 to an orbit with radius $a_2 > a_1$ gives

$$C_1 = -\frac{1}{2} \frac{\mu}{a_1}, \quad C_2 = -\frac{1}{2} \frac{\mu}{a_2},$$

so

$$\Delta C = C_2 - C_1 = \frac{\mu}{2} \left(\frac{1}{a_1} - \frac{1}{a_2} \right). \quad (11.14)$$

If the transfer is affected by an elliptic transfer orbit cotangential to both circular orbits, as shown in figure 11.2, then the operation requires *two impulses*, one at position A and another at position B, where the rocket motor is fired in the opposite direction.

The energy constant of the elliptic transfer orbit, C_T , is given by

$$C_T = -\frac{1}{2} \frac{\mu}{a_T}, \quad (11.15)$$

where a_T is the ellipse's semimajor axis. From figure 11.2 we see that

$$a_T = \frac{1}{2}(a_1 + a_2) \quad (11.16)$$

and therefore

$$C_T = -\frac{\mu}{a_1 + a_2}.$$

The required energy increment at A to place the vehicle into the correct transfer orbit is

$$\Delta C_A = C_T - C_1 = -\frac{\mu}{2a_1} \left(\frac{2a_1 - (a_1 + a_2)}{a_2 + a_1} \right) = \frac{\mu}{2a_1} \left(\frac{a_2 - a_1}{a_2 + a_1} \right). \quad (11.17)$$

Similarly, the energy increment required at B to bring the vehicle from the transfer orbit to the outer circular orbit is

$$\Delta C_B = C_2 - C_T = \frac{\mu}{2a_2} \left(\frac{a_2 - a_1}{a_2 + a_1} \right). \quad (11.18)$$

Those energy changes are due to changes in kinetic energy. Hence

$$\Delta C_A = \frac{1}{2}(v_A + \Delta v_A)^2 - \frac{1}{2}v_A^2 \quad (11.19)$$

and

$$\Delta C_B = \frac{1}{2}v_B^2 - \frac{1}{2}(v_B - \Delta v_B)^2, \quad (11.20)$$

where Δv_A and Δv_B are the changes in velocity at A and B respectively. Equating (11.17) and (11.19) gives

$$\frac{\mu}{2a_1} \left(\frac{a_2 - a_1}{a_2 + a_1} \right) = \frac{1}{2}(v_A + \Delta v_A)^2 - \frac{1}{2}v_A^2.$$

Since $v_A^2 = \mu/a_1$ we have

$$\Delta v_A = \left[\frac{\mu}{2a_1} \frac{a_2 - a_1}{a_2 + a_1} + \frac{1}{2} \frac{\mu}{a_1} \right]^{1/2} - \frac{\mu}{a_1},$$

or

$$\Delta v_A = \left(\frac{\mu}{a_1} \right)^{1/2} \left[\left(\frac{2a_2}{a_2 + a_1} \right)^{1/2} - 1 \right]. \quad (11.21)$$

Similarly, equating (11.18) and (11.20) gives

$$\Delta v_B = \left(\frac{\mu}{a_2} \right)^{1/2} \left[1 - \left(\frac{2a_1}{a_2 + a_1} \right)^{1/2} \right]. \quad (11.21)$$

To calculate the fuel consumption we can use (11.2), because we have ignored drag and gravity losses. For the first impulse we have

$$\Delta v_A = v_e \ln \left(\frac{m_0}{m_A} \right),$$

giving $m_0 - m_A$ (the mass of fuel used) as a function of Δv_A , the exhaust velocity v_e and the mass of the vehicle before operation, m_0 . For the second impulse we have

$$\Delta v_B = v_e \ln \left(\frac{m_A}{m_B} \right),$$

giving $m_A - m_B$ (the mass of fuel used) as a function of Δv_B , v_e and m_A . Combining Δv_A and Δv_B gives the total fuel expenditure

$$\Delta v = \Delta v_A + \Delta v_B = v_e \ln \left(\frac{m_0}{m_B} \right).$$

Fig. 11.3 shows two orbits of radii $a_1 = 1$ and $a_2 = 20$ together with the corresponding elliptic transfer orbit. For the first impulse the rocket has to produce a velocity increment of 0.76π AU/yr (from the circular velocity of 2π AU/yr to the perihelion velocity of the transfer orbit of 2.76π AU/yr). At aphelion of the transfer ellipse it has to accelerate second time to produce a velocity increment of 0.31π AU/yr (from 0.14π AU/yr to 0.45π AU/yr). Thus, the total velocity increment is 1.07π AU/yr.

11.3.2 Parabolic transfer orbits

The velocity increment necessary to bring a body out of the solar system via a parabolic orbit is

$$\Delta v = (\sqrt{2} - 1)v_c,$$

where $v_c = \sqrt{\mu/a_1}$ is the circular velocity; see Eq. (4.81). There is a point where the elliptic transfer orbit becomes as expensive as going onto the parabolic orbit. This is the case for large ratios

$$q = \frac{a_2}{a_1}.$$

We now calculate the ratio, for which

$$\Delta v_A + \Delta v_B = (\sqrt{2} - 1)\sqrt{\frac{\mu}{2a_1}}.$$

Inserting the expressions from above this equation leads to

$$\sqrt{\frac{2q}{1+q}} - 1 + \frac{1}{\sqrt{q}} \left(1 - \sqrt{\frac{2}{1+q}}\right) = \sqrt{2} - 1,$$

or

$$\sqrt{\frac{2q}{1+q}} + \frac{1}{\sqrt{q}} \left(1 - \sqrt{\frac{2}{1+q}}\right) = \sqrt{2},$$

Multiplying by $\sqrt{q}$ leads to

$$q\sqrt{\frac{2}{1+q}} + \left(1 - \sqrt{\frac{2}{1+q}}\right) = \sqrt{2q}.$$

After some manipulations we arrive at the cubic equation

$$8q^3 - 33q^2 + 22q - 1 = 0,$$

which has a positive root at $q = 3.304$. This suggests that for transfers to orbits further away than ~ 3.3 times the original orbit the transfer via a single ellipse may not be the cheapest one. This is confirmed by the example below.

The bi-elliptic transfer

The journey to $a_2 = 20$ AU via a single-elliptic transfer orbit may not necessarily be the cheapest one. A transfer via a bi-elliptic orbit can in some cases be slightly cheaper. Consider a transfer from $a_1 = 1$ AU to $a_2 = 20$ AU via $a_3 = 40$ AU; see Fig 11.4.

We need three impulses. For the first impulse we bring the rocket to an orbit of 40 AU, i.e. we need to accelerate from

$$\sqrt{\frac{\mu}{a_1}} \rightarrow \sqrt{\frac{2\mu}{a_1 + a_3}} \sqrt{\frac{a_3}{a_1}}.$$

The corresponding velocities are given in Fig 11.4. For the second impulse we need to evaluate the velocities at aphelion of the old and the new elliptic transfer orbits, which leads to the velocity increment

$$\sqrt{\frac{2\mu}{a_1 + a_3}} \sqrt{\frac{a_1}{a_3}} \rightarrow \sqrt{\frac{2\mu}{a_2 + a_3}} \sqrt{\frac{a_2}{a_3}}.$$

Finally, to get back onto a circular orbit, but now at $a_2 = 20$ AU, we need the velocity increment

$$\sqrt{\frac{2\mu}{a_2 + a_3}} \sqrt{\frac{a_3}{a_2}} \rightarrow \sqrt{\frac{\mu}{a_2}}.$$

Summing up all those velocity increments we have a *total* velocity increment of 1.05 AU/yr, just somewhat less than for the single-elliptic transfer orbit.

This is just one example of how non-intuitive orbital dynamics can sometimes be. There are more examples, but this is all we have time for.

11.4.5 The fly-past as a velocity amplifier

Assume a space craft is put onto a transfer ellipse from the Earth to Jupiter. When it arrives at Jupiter's orbit it is slower than Jupiter by an amount

$$V = \sqrt{\frac{\mu}{a_2}} - \sqrt{\frac{2\mu}{a_1 + a_2}} \sqrt{\frac{a_1}{a_2}} \quad (11.96)$$

Inserting relevant numbers this amounts to $V = 1.19 \text{ AU/yr}$. Although we haven't derived this expression, we note that the Jovian sphere of influence has a radius given by

$$r = \left(\frac{m}{M}\right)^{2/5} a_2 \approx 0.32 \text{ AU}, \quad (11.97)$$

where we have assumed $m = 1.9 \times 10^{27} \text{ kg}$, $M = 2.0 \times 10^{30} \text{ kg}$, and $a_2 = 5.2 \text{ AU}$. Here, m and M are the masses of Jupiter and the Sun, respectively, and a_2 is the semimajor axis of Jupiter's orbit. Inside the sphere of influence we can ignore the effects of the Sun. The space craft enters the sphere at speed V in the opposite direction relative to the orbital motion of Jupiter. As viewed from inside Jupiter's sphere of influence the space craft appears to have come from infinity at that speed V . The space craft will therefore enter a hyperbolic orbit around Jupiter. This also means that the space craft will exit Jupiter's sphere of influence with the same speed V , but only in another direction.

At first glance this may not appear to be of not much use, because we have learned that the value of the semimajor axis does not change if the speed at some point is the same and just its direction different. This is however not correct in this case, because all this happens relative to the motion of Jupiter. Now, suppose that the encounter changes the trajectory of the space craft by nearly 180 degrees, then it will exit Jupiter's sphere of influence in the direction of the motion of Jupiter itself. This means that the speed of the space craft is now the sum of V and of Jupiter's orbital speed. We shall see in a moment that this does indeed exceed the escape velocity of the solar system at that distance from the Sun.

We now need to calculate the opening angle of the hyperbola. We do this by assuming that the space craft if brought close enough to Jupiter that it will pass by with a perijove of r_p which is comparable to Jupiter's radius r_J . (We will however consider different values of r_p/r_J .)

Remember, for a hyperbolic orbit the parameter p is equal to $p = a(e^2 - 1)$ and therefore the distance at perijove is equal to

$$r_p = \frac{p}{1 + e} = a \frac{1 - e^2}{1 + e} = a(1 - e).$$

We also have

$$b = a\sqrt{e^2 - 1}. \quad (11.99)$$

The asymptotes of the hyperbolic encounter are given by

$$\tan \psi = \frac{b}{a} \quad (11.100)$$

The opening angle is 2ψ , and therefore the change in the trajectory θ is given by $2\psi + \theta = \pi$, which means

$$\theta = 2 \tan^{-1} \frac{1}{\sqrt{e^2 - 1}}. \quad (11.100)$$

In the following table we have listed the resulting values of e and θ for different values of r_p/r_J . Clearly, only values of r_p that are less than the radius of Jupiter's sphere of influence make sense. Hence, values of r_p/r_J larger than about 66 are no longer meaningful.

In order to calculate the value of the semimajor of the space craft's orbit around Jupiter's we use Eq. (4.37), ie

$$a = \frac{\mu_J}{(2\mu_J/r) - V^2} = 0.032 \text{ AU} \quad (4.37)$$

where we put

$$\mu_J = Gm = GM(m/M) = \mu(m/M) = 0.0375,$$

and r is the radius of Jupiter's sphere of influence.

The idea is to bring the space craft as close as possible to Jupiter, without of course crashing into it's surface. So we assume that the perijove of the space craft, r_p , is close to Jupiter's radius, r_J , or possibly a few times r_J . Now that we know a we can, for a given value of r_p , calculate the eccentricity. We do this by using the

equation for conical sections, applied to an angle $f = 0$, so $r_p = p/(1 + e)$, using $p = a(e^2 - 1)$ for a hyperbolic orbit. Thus, we have $r_p = a(e^2 - 1)/(1 + e) = a(e - 1)$, so

$$e = r_p/a + 1$$

For $r_p = r_J = 0.000477$ AU we have $e = 1.015$, which is very close to the value for a parabolic orbit.

We now want to calculate the change in the angle of the trajectory. Since the space craft is slower than Jupiter itself, it approaches Jupiter from the front. If the eccentricity is very close to unity the opening angle of the hyperbola, 2ψ , would be close to 360 degrees, so the space craft would be tossed around Jupiter by almost 180 degrees, meaning that it would then be along the direction of Jupiter. For values of e different from unity the angle ψ (half-opening angle) is given by Eq. (4.89), so

$$\cos \psi = -\frac{1}{e}.$$

The angle by which the trajectory of the space craft changes is $\theta = \pi - 2\psi$, so

$$\theta = 2 \cos^{-1} \left(\frac{1}{e} \right).$$

The table below gives the values of θ for different values of r_p/r_J . Just for curiosity we have given the results for up to $r_p/r_J = 200$, even though the space craft would then already be way out of Jupiter's sphere of influence.

r_p/r_J	r_p	e	θ [degree]
1	0.00048	1.015	160.2
2	0.00095	1.030	152.2
5	0.00238	1.075	136.9
10	0.00477	1.151	120.7
20	0.00954	1.301	100.5
50	0.02385	1.753	69.6
100	0.04770	2.506	47.0
200	0.09540	4.011	28.9

Figure 11.1: Illustration of how a car is accelerated by throwing cannon balls of mass dm and velocity v . This leads to a change of momentum of the car by the amount mdv , where m is the mass of the car and dv the increase in speed.

Figure 11.2: Speed increase of a rocket as a function of the fuel consumption normalised to the mass of the payload. In the second plot the scale on the abscissa is logarithmic.

Figure 11.3: Elliptic transfer orbit

Figure 11.4: Bi-elliptic transfer orbit