

Fluid Flow and Cosmical Fluids

Lecture notes to MAS371

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This part of the course is a continuation of MAS218. The main difference is that here we focus more on mathematical aspects and on solving the equations of fluid dynamics (and magnetohydrodynamics). We begin by reviewing the basic equations and summarising the units used and some important physical quantities.

The present notes are still in a preliminary state. Some parts is covered more completely than other parts, but this does not mean they are more important.

Units and physical constants

Throughout this course we adopt SI units. However, most common in astrophysics are actually gaussian cgs units. In order to facilitate comparison with the literature we sometimes also give some results in cgs units.

Table 1: Table of useful physical constants.

Quantity	Symbol	value	units
Newton's constant	G	6.673×10^{-11}	$\text{m}^3 \text{kg}^{-1} \text{s}^{-2}$
Stefan-Boltzmann constant	σ	5.67×10^{-8}	$\text{kg s}^{-3} \text{K}^{-4}$
Universal gas constant	$\mathcal{R}$	8315	$\text{m}^2 \text{s}^{-2} \text{K}^{-1}$
Speed of light	c	3×10^8	m s^{-1}
Induction constant	μ_0	$4\pi \times 10^{-7}$	$\text{G m s}^2 \text{kg}^{-1}$

In addition to SI units we also use sometimes units that are appropriate in astrophysical context. For example distances in the solar system and in protostellar discs are often measured in astronomical units (1 AU is the average distance between the sun and the earth). On galactic scales one often measures distances in parsec or in kiloparsec. (The distance between the sun and the centre of the Milky Way is approximately 10 kpc.) Time is sometimes convenient to measure in years, and mass is sometimes expressed in units of solar masses ($M_\odot$).

Table 2: Some useful units used in astrophysics and the corresponding conversion into SI units.

1 AU	=	$1.5 \times 10^{11} \text{ m}$
1 pc	=	$3.1 \times 10^{16} \text{ m}$
$1M_\odot$	=	$2.0 \times 10^{30} \text{ kg}$
1 yr	=	$3.2 \times 10^7 \text{ s}$
1 Myr	=	10^6 yr
1 Mm	=	10^6 m

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Chapter 1

Introduction

Basic equations of fluid dynamics

There are two important equations and will be used throughout this course: the continuity equation and the momentum equation. We write them in the form

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \quad (1.1)$$

and

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{1}{\rho} \nabla p - \nabla \Phi + \mathbf{f}, \quad (1.2)$$

respectively. Here, ρ and $\mathbf{v}$ are density and velocity, respectively. The other quantities are pressure p , gravity (or any other) potential Φ , and external body force (per unit mass) $\mathbf{f}$, such as viscous force and Lorentz force. The gravitational potential is

$$\Phi = -\frac{GM}{r}, \quad (1.3)$$

where M is the central mass, G is the gravitational constant, and r is the distance from the centre of mass (centre of the sun in the solar wind problem that we are going to consider first).

There are other ways of writing the basic equations, for example in ‘conservative form’ or using the lagrangian derivative, but we’ll come to that later.

Equations (1.1) and (1.2) need to be supplemented by an equation of state to determine the pressure in terms of density and temperature T or entropy density s . In the majority of cases, however, we assume that the gas is polytropic, $p = p(\rho)$, and often we assume that it is isothermal, i.e. $p = c_s^2 \rho$. In that case eq. (1.2) takes the form

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -c_s^2 \nabla \ln \rho - \nabla \Phi + \mathbf{f}, \quad (1.4)$$

There is a number of other important physical effects that would lead to additional forces in this equation. We mention here two examples, magnetic fields and rotation. For deeper explanations and

derivations we refer to the relevant chapters. Magnetic fields lead to the Lorentz force,

$$\mathbf{F}_{\text{Lor}} = \mathbf{J} \times \mathbf{B} / \rho, \quad (1.5)$$

where $\mathbf{B}$ is the magnetic field and $\mathbf{J}$ the current density. Another example is overall rotation, which leads to the Coriolis force,

$$\mathbf{F}_{\text{Cor}} = -2\boldsymbol{\Omega} \times \mathbf{v}, \quad (1.6)$$

where $\boldsymbol{\Omega}$ is the angular velocity and points in the direction normal to the rotation plane. For example, if the rotation axis is in the z -direction, i.e. $\boldsymbol{\Omega} = (0, 0, \Omega)^T$, we have

$$\mathbf{F}_{\text{Cor}} = \begin{pmatrix} 0 \\ 0 \\ -2\Omega \end{pmatrix} \times \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \begin{pmatrix} +2\Omega v_y \\ -2\Omega v_x \\ 0 \end{pmatrix}. \quad (1.7)$$

In the case of magnetic fields, the field itself satisfies an evolution equation,

$$\frac{\partial \mathbf{B}}{\partial t} = \boldsymbol{\nabla} \times (\mathbf{v} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}, \quad (1.8)$$

where the magnetic diffusivity η is assumed to be constant. This equation follows from the Maxwell's equations and Ohm's law.

Text books

Some useful text books include:

Choudhuri, A. R. 1998 *The Physics of Fluids and Plasmas*. Cambridge University Press.

Shore, S. N. 1992 *An introduction to astrophysical hydrodynamics*. Academic Press.

Shu, F. H. 1992 *The physics of astrophysics. Vol. II. Gas dynamics*. University Science Books, Sausalito, California.

Zeilik, M. 1994 *Astronomy: the evolving universe*. John Wiley & Sons.

The book by Zeilik has many pictures and explains things qualitatively (there are hardly any formulae), but is otherwise not directly relevant to this course. The book by Shore is more mathematical and interesting to read. It has many misprints however. Finally, the book by Shu covers things in much more detail and is generally very good.

Chapter 2

Winds and outflows

Astrophysical flows are usually highly compressible. There are some specific effects resulting from that, one of which concerns the formation of winds and other types of outflows. The general principle of how a flow can attain supersonic speeds in a smooth manner (without going through a shock) can best be explained with the example of the Laval nozzle that is used in all rocket motors where the exhaust velocity can be 2000...3000 m/s, which is up to ten times the speed of sound.

We follow here the book by Shore, Chapter 8, p 257. We summarise what's written there, and add some details that help to understand the problem. We begin this section by first discussing the general issue of how a flow can be made supersonic. We then apply this to winds and outflows from stars and discs and mention other circumstances where this formalism can be used. A rather important application is jets, i.e. outflows from accretion discs that become highly collimated. The collimation is probably due to the magnetic field, but this is very much an open research topic at the moment. The highly relativistic version of the wind problem is relevant for understanding outflows from active galactic nuclei (AGNs) and the gamma-ray burst (GRB) phenomenon.

2.1 The analogy with the Laval nozzle

We are used to think that the flow through a pipe is becoming faster when the cross-section becomes smaller (garden hose experiment!). However, this is only true when the fluid is incompressible. That's often not a good approximation in astrophysics. The following problem illustrates this.

In a steady state the mass flux through a pipe is conserved, ie

$$\rho v_x S = \text{const.} \quad (2.1)$$

Here, $S = S(x)$ is the cross-section, $v_x(x)$ the streamwise velocity and $\rho(x)$ the density. This equation follows from integrating the equation $\nabla \cdot (\rho \mathbf{v}) = 0$ over the surface of a pipe that becomes narrower, and by applying Gauss' divergence theorem; see figure 2.1.

Indeed, if ρ is constant, (2.1) shows that a larger cross-sectional area S implies a smaller speed v_x .

But what if ρ actually decreases? In the following we will see an example where supersonic speeds can be achieved by decreasing density and increasing the cross-sectional area in a suitable manner.

The velocity is obtained by solving the steady, one-dimensional, isothermal momentum equation without any extra forces,

$$v_x \frac{dv_x}{dx} = -c_s^2 \frac{d \ln \rho}{dx}. \quad (2.2)$$

We now use eq. (2.1), differentiate logarithmically, i.e.

$$0 = \frac{1}{\rho v_x S} \frac{dv_x}{dx} (\rho v_x S) = \frac{d}{dx} \ln(\rho v_x S) = \frac{d \ln \rho}{dx} + \frac{d \ln |v_x|}{dx} + \frac{d \ln S}{dx}, \quad (2.3)$$

so we have

$$\frac{d \ln \rho}{dx} = -\frac{d \ln |v_x|}{dx} - \frac{d \ln S}{dx}. \quad (2.4)$$

Using that in eq. (2.2) yields

$$v_x \frac{dv_x}{dx} = -c_s^2 \left(-\frac{d \ln |v_x|}{dx} - \frac{d \ln S}{dx} \right). \quad (2.5)$$

We move the first term on the rhs to the left and on the left we write

$$v_x \frac{dv_x}{dx} = v_x^2 \frac{d \ln |v_x|}{dx}. \quad (2.6)$$

This yields the equation

$$(v_x^2 - c_s^2) \frac{d \ln |v_x|}{dx} = c_s^2 \frac{d \ln S}{dx}, \quad (2.7)$$

where $S(x)$ is the cross-sectional area of the nozzle, which is a known function of x . To obtain $v_x(x)$ we can integrate

$$\frac{d \ln |v_x|}{dx} = c_s^2 \frac{d \ln S/dx}{v_x^2 - c_s^2}, \quad (2.8)$$

but in order for the solution to be regular when $|v_x| = c_s$ we have to require that the nominator vanishes at the same point. Thus, the critical point is where $S(x)$ has a minimum, because then $d \ln S/dx = 0$, which must be where $v_x = c_s$. Define $\rho v_x S = \dot{M}$ and integrate to obtain an implicit equation for $v(x)$:

$$\frac{1}{2} v_x^2 - c_s^2 \ln |v_x| - c_s^2 \ln S = \mathcal{E} - c_s^2 \ln \dot{M} \equiv \mathcal{E}'. \quad (2.9)$$

This is just the usual Bernoulli equation that we have encountered elsewhere. We can now determine $\mathcal{E}'_{\text{crit}}$ by applying the known values of r and v_r at the critical. For $c_s = \dot{M} = 1$ we find $\mathcal{E}'_{\text{crit}} = \frac{1}{2} c_s^2 - c_s^2 \ln S_{\text{min}}$.

2.2 The isothermal wind problem

We now consider the steady, isothermal wind problem. We adopt spherical polar coordinates, (r, θ, ϕ) , but assume spherical symmetry in this case, so $\partial/\partial\theta = \partial/\partial\phi = 0$. The continuity and Euler equations are then

$$\frac{1}{r^2} \frac{d}{dr} (r^2 \rho v_r) = 0, \quad (2.10)$$

$$v_r \frac{dv_r}{dr} = c_s^2 \frac{d \ln \rho}{dr} - \frac{GM}{r^2}. \quad (2.11)$$

These equations can be brought into the form

$$(v_r^2 - c_s^2) \frac{d \ln |v_r|}{dr} = \frac{2c_s^2}{r} - \frac{GM}{r^2}. \quad (2.12)$$

There is a critical point r_* where the flow becomes transonic. At that point the left hand side and the right hand side of the equation must vanish simultaneously, ie

$$\frac{2c_s^2}{r} = \frac{GM}{r^2}. \quad (2.13)$$

This yields for the value of the critical point

$$r_* = \frac{GM}{2c_s^2} \quad (2.14)$$

Assuming $c_s = 100 \text{ km/s}$, and using $G \approx 7 \times 10^{-11} \text{ m}^3 \text{ s}^{-2} \text{ kg}^{-1}$, $M = 2 \times 10^{30} \text{ kg}$, $1 \text{ AU} = 1.5 \times 10^{11} \text{ m}$, we have $r_* \approx 0.05 \text{ AU}$.

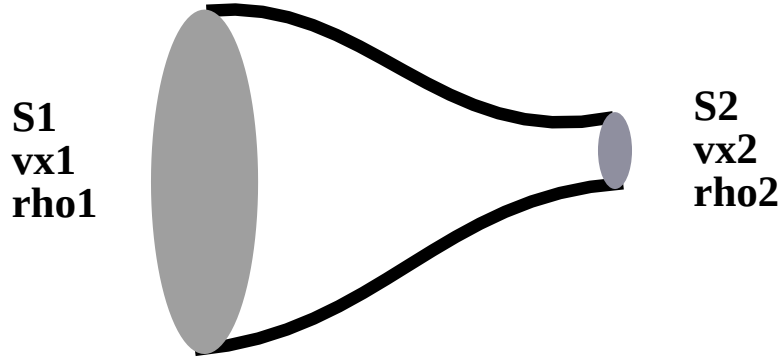


Figure 2.1: In the steady case we have $\nabla \cdot \rho \mathbf{v} = 0$. Using Gauss' divergence theorem it follows that $\oint \rho \mathbf{v} \cdot d\mathbf{S} = 0$. The two surfaces, S_1 and S_2 , are the only places where the $\mathbf{v} \cdot d\mathbf{S} \neq 0$.

The steady wind problem is characterised by two integrals of motions:

$$\frac{1}{2}v_r^2 + c_s^2 \ln \rho - \frac{GM}{r} = \mathcal{E}, \quad (2.15)$$

$$\dot{M} = 4\pi r^2 \rho v_r, \quad (2.16)$$

see Eq. (9) in Shore's book. From this we get

$$\frac{1}{2}v_r^2 - c_s^2 \ln v_r - 2c_s^2 \ln r - \frac{GM}{r} = \mathcal{E}' \quad (2.17)$$

where $\mathcal{E}' = \mathcal{E} - c_s^2 \ln(\dot{M}/4\pi)$. (Note that the $\mathcal{C}$ in Shore's book changes its meaning all the time).

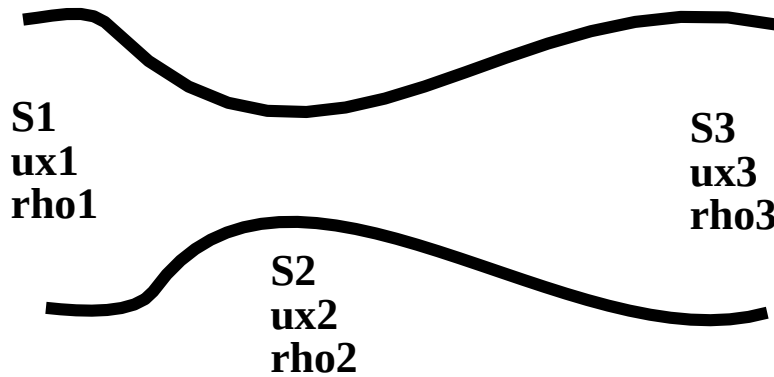


Figure 2.2: Cross-sectional area of a laval nozzle. If the flow is able to reach the sound speed at the point of minimal cross-section, it must go supersonic behind that point.

To find the energy for the critical solution we assume a critical radius r_* , so $c_s = \sqrt{GM/2r_*}$. At $r = r_*$ we have $v = c_s$. This yields $\mathcal{E}$ for the critical solution that goes through the critical point:

$$\mathcal{E}'_{\text{crit}} = \frac{1}{2}c_s^2 - c_s^2 \ln c_s - 2c_s^2 \ln r_* - \frac{GM}{r_*} \quad (2.18)$$

We may find the solution through iteration:

$$v_{\text{new}} = \sqrt{2(\mathcal{E}_r + c_s^2 \ln v)} \quad (2.19)$$

where $\mathcal{E}_r \equiv \mathcal{E}' + 2c_s^2 \ln r_* + GM/r_*$ is introduced for convenience. As usual, we can get another solution from the inverse iteration formula

$$v_{\text{new}} = \exp\left(\frac{1}{2}v^2 - \mathcal{E}_r\right) \quad (2.20)$$

For $\mathcal{E}' < \mathcal{E}'_{\text{crit}}$ the upper solutions are obtained from eq. (2.19) whilst the lower solutions are obtained by iterating eq. (2.20). The critical solution, as well as two more solutions are shown in figure 2.3.

Another possibility of obtaining a graphical representation of the possible wind solutions is to do a contour plot of $\mathcal{E}$ as a function of r and v_r ; see figure 2.4.

Note that the iteration process always yields that solution that has a minimum or a maximum at $r = r_*$. To get the physically sensible solution one needs to reconnect the right branches by hand. This is done in figure 2.5, where we have plotted the critical solution together with the density profile.

2.3 The time-dependent wind problem

It is illuminating to consider the time-dependent problem. For example, one may wonder what happens when one were to increase the density or the velocity at the bottom. The governing equations are

$$\frac{\partial \ln \rho}{\partial t} = -v_r \frac{\partial \ln \rho}{\partial r} - 2 \frac{v_r}{r} - \frac{\partial v_r}{\partial r} \quad (2.21)$$

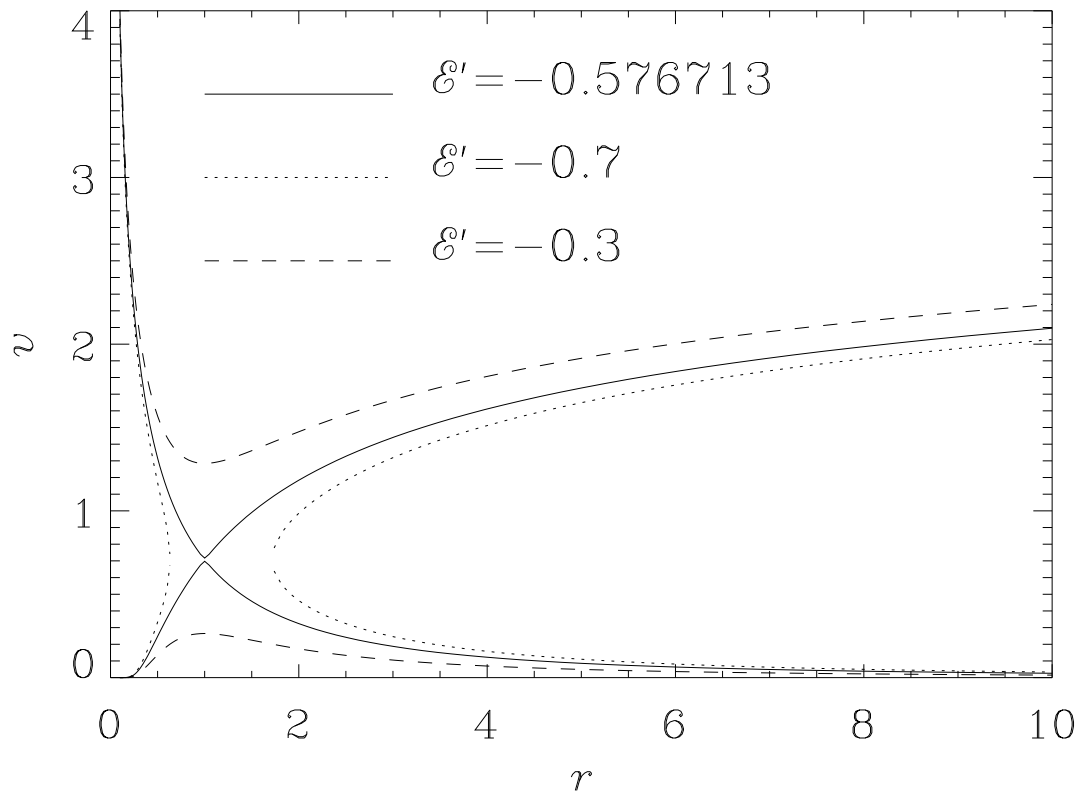


Figure 2.3: The critical wind solution, together with two other solutions for different values of $\mathcal{E}'$. Here we use $r_* = 1$, $GM = 1$, and so $c_s^2 = 0.5$.

$$\frac{\partial v_r}{\partial t} = -v_r \frac{\partial v_r}{\partial r} - c_s^2 \frac{\partial \ln \rho}{\partial r} - \frac{GM}{r^2}. \quad (2.22)$$

Since the flow comes from the inner boundary at $r = r_0$ we have to specify boundary conditions at $r = r_0$. Video animations of numerical experiments show that specifying the value of v_r at the lower boundary is inconsistent. The solution has a strong desire to come back to the steady solution discussed above. On the other hand, the density on the lower boundary may well be specified arbitrarily. The solution has then just another value of $\dot{M}$.

The solution can actually be reversed and then we have the problem of spherically symmetric accretion (Bondi accretion). This is the reason why one should always write $\ln |v_r|$.

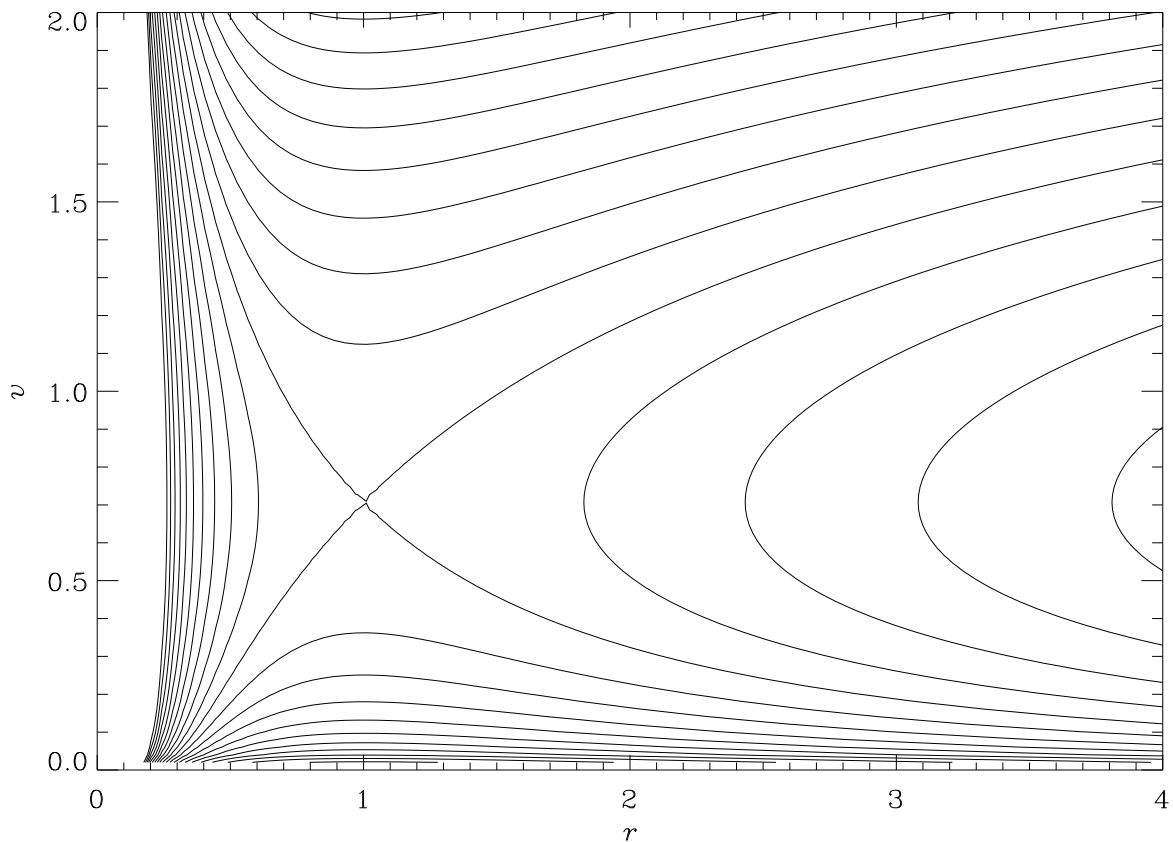


Figure 2.4: Contour plot of $\mathcal{E}'$. The contour levels are equidistant in steps of 0.2, symmetrically about the critical value $\mathcal{E}' = -0.576713$. Again, $r_* = 1$, $GM = 1$, and so $c_s^2 = 0.5$.

The trick is to write the equations in the form $(v_r^2 - c_s^2)d \ln |v_r|/dx = \text{something}$. This something then tells us *where* the critical point is. Once we know that, we plug that back into the integrated momentum equations to obtain an implicit equation for v_r that goes through the critical point.

2.4 Roche-lobe overflow

There are many semi-detached binary stars where mass flows from the secondary to the primary. Somewhere between the two stars there is the L_1 point, which is an equilibrium point where the forces from both stars and the centrifugal force all balance (see the box). Near that point the gravitational potential Φ can be approximated by

$$\Phi = \Phi_0 - \frac{1}{2}\Phi_2 x^2, \quad (2.23)$$

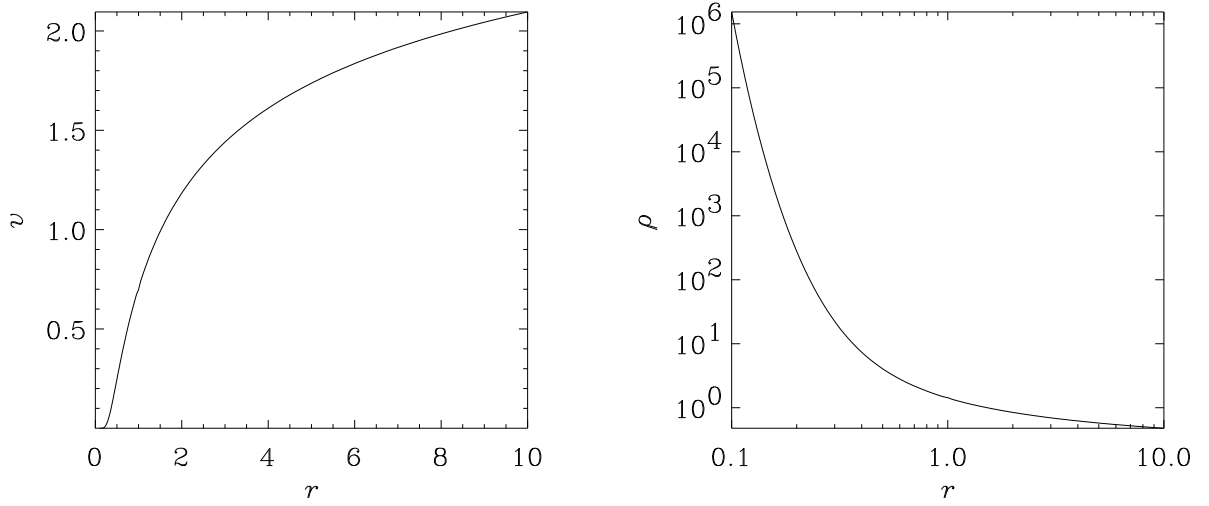


Figure 2.5: Critical solution for v and ρ (assuming $\dot{M} = 4\pi$).

where x is the coordinate in the direction from the secondary to the primary; see figure 2.6. A derivation is given in the Appendix.

Assume that the Coriolis force can be neglected, and that the flow across the point $x = 0$ can be described by the stationary, the isothermal, one-dimensional fluid equations take the form

$$\frac{d}{dx}(\rho v_x) = 0, \quad (2.24)$$

$$v_x \frac{dv_x}{dx} = -c_s^2 \frac{d \ln \rho}{dx} - \frac{d\Phi}{dx}, \quad (2.25)$$

where $c_s = \text{const}$ is the speed of sound.

We rewrite the continuity equation in the form

$$\frac{d \ln \rho}{dx} = -\frac{d \ln |v_x|}{dx}, \quad (2.26)$$

plug this into the momentum equation

$$v_x \frac{dv_x}{dx} = -c_s^2 \frac{d \ln \rho}{dx} - \frac{d\Phi}{dx}, \quad (2.27)$$

and use $v_x \frac{dv_x}{dx} = v_x^2 \frac{d \ln |v_x|}{dx}$ to obtain

$$(v_x^2 - c_s^2) \frac{d \ln |v_x|}{dx} = -\frac{d\Phi}{dx}, \quad (2.28)$$

or

$$(v_x^2 - c_s^2) \frac{d \ln |v_x|}{dx} \approx -2\Phi_2(x - x_{L1}). \quad (2.29)$$

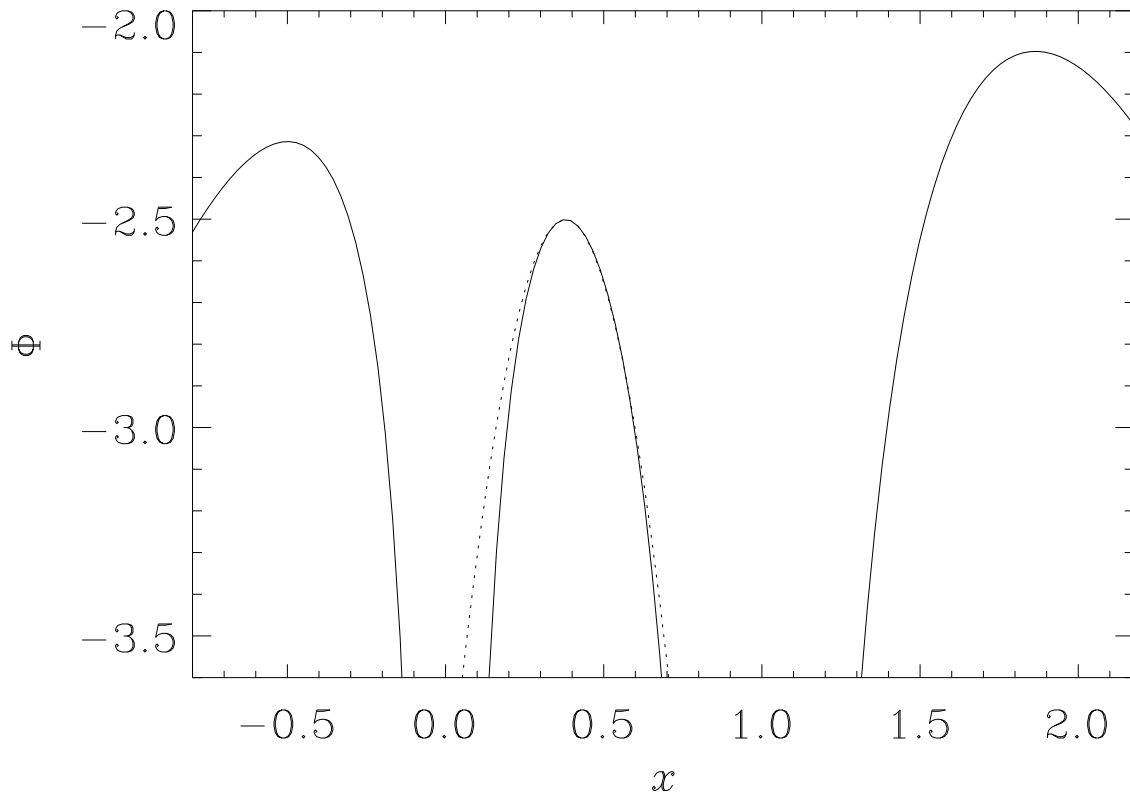


Figure 2.6: The Roche potential for a binary with masses $M_1 = 1M_\odot$ and $M_2 = 0.3M_\odot$. The primary is located at $x = 0$ and the secondary at $x = 1$. The dotted line gives the Taylor expansion used in the present analysis.

This shows that the flow becomes supersonic at the L_1 point. This is therefore automatically also the critical point. To find an implicit algebraic equation for v_x we integrate once and use $\rho v_x = \text{const} = \dot{M}$

$$\frac{1}{2}v_x^2 - c_s^2 \ln |v_x| - \frac{1}{2}\Phi_2 x^2 = \mathcal{E} - c_s^2 \ln \dot{M} - \Phi_0 \equiv \mathcal{E}', \quad (2.30)$$

where $\mathcal{E}$ and $\mathcal{E}'$ are integration constants. At the critical point we have

$$\mathcal{E}'_{\text{crit}} = \frac{1}{2}c_s^2 - c_s^2 \ln c_s. \quad (2.31)$$

We don't know the values of c_s and $\dot{M}$. For the purpose of the following discussion let's assume $c_s = \dot{M} = 1$. This gives $\mathcal{E}'_{\text{crit}} = 1/2 - 2.5 = -2$. In the following plot we give the graph of the critical solution and compare with solutions for $\mathcal{E}' = -1$ (dotted line) and $\mathcal{E}' = -3$ (dashed line). For $\mathcal{E}' < \mathcal{E}'_{\text{crit}}$ there are no solutions at the L_1 point, so this will not be a physical solution. For $\mathcal{E}' > \mathcal{E}'_{\text{crit}}$ the solution never goes through the sonic point either and stays all the time supersonic or subsonic.

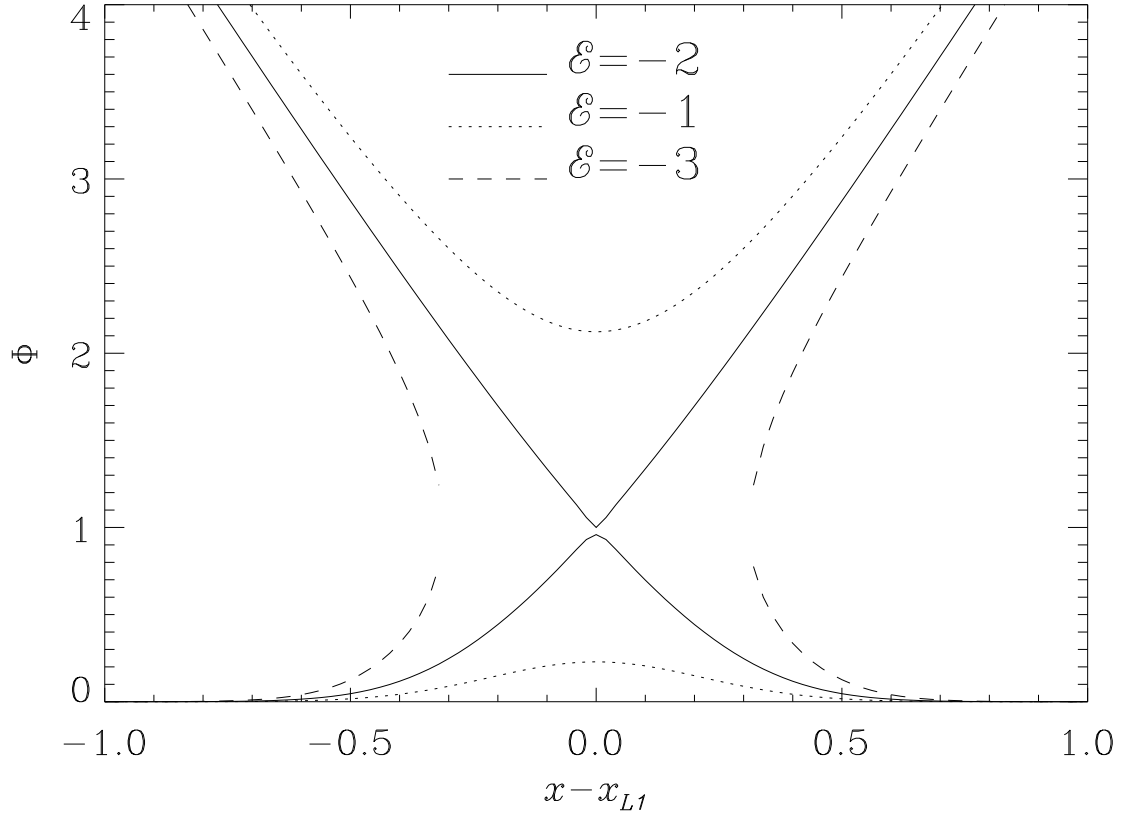


Figure 2.7: Solutions for the Roche lobe overflow.

We now proceed and rewrite eq. (2.30) by substituting the integration constant using eq. (2.31), i.e.

$$\frac{1}{2}v_x^2 - c_s^2 \ln |v_x| - \frac{1}{2}\Phi_2 x^2 = \frac{1}{2}c_s^2 - c_s^2 \ln c_s. \quad (2.32)$$

or

$$\frac{1}{2}(v_x^2 - c_s^2) - c_s^2 \ln(|v_x|/c_s) - \frac{1}{2}\Phi_2 x^2 = 0. \quad (2.33)$$

This implicit equation can be solved using the approximation

$$\ln(|v_x|/c_s) \approx (|v_x|/c_s) - 1 \quad (2.34)$$

near the critical point, $(|v_x|/c_s) \approx 1$. We thus obtain a quadratic equation,

$$\frac{1}{2}(v_x^2 - c_s^2) - c_s |v_x| + c_s^2 - \frac{1}{2}\Phi_2 x^2 = 0, \quad (2.35)$$

or

$$v_x^2 - 2c_s |v_x| + c_s^2 - \Phi_2 x^2 = 0. \quad (2.36)$$

There are two solutions to this quadratic equation,

$$|v_x|_{1,2} = c_s \pm \sqrt{c_s^2 - c_s^2 + \Phi_2 x^2}, \quad (2.37)$$

or

$$|v_x|_{1,2} = c_s \pm \sqrt{\Phi_2} x. \quad (2.38)$$

Near the critical point the pressure gradient is given by

$$c_s^2 \frac{d\rho}{dx} = -\rho \left(v_x \frac{dv_x}{dx} + \frac{d\Phi}{dx} \right), \quad (2.39)$$

or

$$c_s^2 \frac{d\rho}{dx} = -\rho \left(c_s \sqrt{\Phi_2} - \Phi_2 \right), \quad (2.40)$$

or, because $|x| \ll 1$,

$$c_s^2 \frac{d\rho}{dx} \approx -\rho c_s \sqrt{\Phi_2}. \quad (2.41)$$

Assuming that $\Phi_2 = 25 \text{ s}^{-2}$ and $c_s = 10 \text{ km/s}$, the value of the acceleration, $-c_s^2 d \ln \rho / dx$, at the critical point is around $50 \text{ km s}^{-2} = 5 \times 10^4 \text{ m s}^{-2}$. The acceleration GM/R^2 of a single star with $M = 2 \times 10^{30} \text{ kg}$ at distance $R = 10^7 \text{ m}$ is around

$$\frac{GM}{R^2} = \frac{6.67 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2} \times 2 \times 10^{30} \text{ kg}}{10^{14} \text{ m}^2} = 1.4 \times 10^6 \text{ m s}^{-2}, \quad (2.42)$$

which is about 30 times larger than acceleration from the pressure gradient at the critical point.

2.5 What drives the wind? The need for a nonstatic corona

The following consideration illustrates that there cannot be a static solar corona. In the solar corona heat is transported via conduction (as opposed to radiation or convection). In the corona (Spitzer-type) heat conduction is important, so the conductivity K satisfies

$$K = K_0 \left(\frac{T}{T_0} \right)^{5/2}. \quad (2.43)$$

The heat flux is then given by

$$\mathbf{F} = -K \nabla T. \quad (2.44)$$

In order to have a steady state (vanishing divergence of the heat flux, i.e. $\nabla \cdot \mathbf{F} = 0$) we have to require that the temperature satisfies

$$T = T_0 \left(\frac{r}{r_0} \right)^{-2/7}, \quad (2.45)$$

where T_0 is the temperature at the base of the corona at $r = r_0$. This is significantly shallower than the temperature for a polytropic stratification, which would imply a r^{-1} behaviour with

$$T_{\text{poly}} = \left(1 - \frac{1}{\gamma} \right) \frac{\mu}{\mathcal{R}} \frac{GM}{r}. \quad (2.46)$$

Assuming a perfect gas, i.e. $p = \mathcal{R}T\rho$ one can show that

$$p = p_0 \exp \left\{ \frac{7r_0}{5H_0} \left[\left(\frac{r}{r_0} \right)^{-5/7} - 1 \right] \right\}. \quad (2.47)$$

Here p_0 is the pressure at the base of the corona and H_0 is the scale height of the pressure, i.e. the distance over which the pressure changes by a factor of e . The problem with this expression is that at infinity, $r \rightarrow \infty$, the pressure remains finite,

$$p(\infty) = p_0 \exp \left[-\frac{7r_0}{5H_0} \right]. \quad (2.48)$$

In fact, for the sun H_0 is comparable with the solar radius, r_0 , and so $p(\infty)$ is just somewhat smaller than p_0 , which is definitely quite unrealistic. For a correct solution the pressure must become comparable with the interstellar pressure, which is orders of magnitude smaller than the pressure near the sun. This inconsistency leads then to the idea of a nonstatic corona with an outflow.

Chapter 3

Energy conservation and shocks

In the previous chapter we have seen that a flow can go smoothly from subsonic to supersonic. In the majority of cases, however, this happens more violently and a shock usually forms somewhere in the flow. A shock is a discontinuity in the fluid dynamical variables. In a shock kinetic energy is converted into thermal energy such that the *total* energy is conserved. (This is different however in radiative shocks, where the energy is radiated away.) To proceed we first need to derive the energy equation.

3.1 Derivation of the energy equation

The first law of thermodynamics states that

$$TdS = dE + pdV, \quad (3.1)$$

where T is the temperature, S the entropy, E the internal energy, p the pressure and V the volume. For later reference we mention here that TdS can also be expressed as

$$TdS = dH - Vdp, \quad (3.2)$$

where H is enthalpy. This expression can be important when rewriting the pressure gradient term in the momentum equation. In fluid dynamic we measure those quantities with respect to some unit mass. Thus, we have

$$Tds = de + pdv, \quad (3.3)$$

where s is the specific entropy per unit mass, e is the specific internal energy per unit mass, and v is the specific volume per unit mass (not to be confused with the velocity). For a perfect gas the temperature is related to e via $e = c_v T$, where c_v is the specific heat for constant volume. The specific volume per unit mass is just the ordinary fluid density, so $v = 1/\rho$. Since $dv = d(\rho^{-1}) = -\rho^{-2}d\rho = -\rho^{-1}d \ln \rho$ we have

$$Tds = de - \frac{p}{\rho} d \ln \rho. \quad (3.4)$$

In the following we shall always work with the density and never with the specific volume. Hence, from now on, v shall always refer to velocity. In fluid dynamics we are interested in the total time derivative, so we have

$$T \frac{Ds}{Dt} = \frac{De}{Dt} - \frac{p}{\rho} \frac{D \ln \rho}{Dt}. \quad (3.5)$$

Now, from the continuity equation we have

$$0 = \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = \frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho + \rho \nabla \cdot \mathbf{v} = \frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{v}, \quad (3.6)$$

or, dividing by ρ ,

$$\frac{D \ln \rho}{Dt} = -\nabla \cdot \mathbf{v}. \quad (3.7)$$

Using this in (3.5) yields

$$T \frac{Ds}{Dt} = \frac{De}{Dt} + \frac{p}{\rho} \nabla \cdot \mathbf{v}. \quad (3.8)$$

In an adiabatic fluid the specific entropy is constant, ie $Ds/Dt = 0$. Hence, the evolution of e is given by

$$\frac{De}{Dt} = -\frac{p}{\rho} \nabla \cdot \mathbf{v}. \quad (3.9)$$

This equation shows that if there is a compression, then $\nabla \cdot \mathbf{v} < 0$ and so e increases. This effect is known as compressional heat. Conversely, if there is an expansion then $\nabla \cdot \mathbf{v} > 0$ and e decreases. This effect is utilised in every fridge.

3.2 Sound waves

Shocks may occur in situations where the velocity is supersonic. This is typically when the Mach number, $\text{Ma} = |\mathbf{v}|/c_s$, exceeds unity. Shocks can form when the amplitude of sound waves becomes comparable to the sound speed, so that the Mach number becomes of order unity. Since the sound speed is proportional to the temperature, this may occur when the gas cools, or when the wave reaches a region where the temperature is low.

In the previous chapter on winds we have seen, however, that the formation of a shock is not compulsory and that smooth transonic transitions are quite possible. Conversely, a shock may also occur under subsonic conditions if one just waits long enough for a shock wave to pile up.

In the linear regime, ie $\text{Ma} \ll 1$, the nonlinear terms $\mathbf{v} \cdot \nabla$ can be neglected and the governing equations are then, in 1 dimension,

$$\frac{\partial \ln \rho}{\partial t} = -\frac{\partial v_x}{\partial x} \quad (3.10)$$

$$\frac{\partial v_x}{\partial t} = -c_s^2 \frac{\partial \ln \rho}{\partial x} \quad (3.11)$$

Differentiating the second equation in time and substituting the first one for $\ln \rho$ yields the wave equation,

$$\frac{\partial^2 v_x}{\partial t^2} = c_s^2 \frac{\partial^2 v_x}{\partial x^2}, \quad (3.12)$$

which permits solutions of the form

$$\ln \rho = \text{Ma} \cos k(x \mp c_s t), \quad (3.13)$$

$$v_x = \pm c_s \text{Ma} \cos k(x \mp c_s t), \quad (3.14)$$

where the upper and lower signs correspond to forward and backward propagating waves, respectively. (Here we have ignored the possibility of an arbitrary phase shift.)

3.3 Fluid equations in conservative form

In one dimension the ideal fluid equations can be written in conservative as form

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v_x) = 0, \quad (3.15)$$

$$\frac{\partial}{\partial t}(\rho v_x) + \frac{\partial}{\partial x}(\rho v_x^2 + p) = 0, \quad (3.16)$$

$$\frac{\partial}{\partial t}(\frac{1}{2}\rho v_x^2 + \rho e) + \frac{\partial}{\partial x}[v_x(\frac{1}{2}\rho v_x^2 + \rho e + p)] = 0. \quad (3.17)$$

where e is the internal energy density per unit mass, and the other variables have their usual meaning.

3.4 Shocks

In a frame of reference comoving with the shock the shock is stationary. Therefore $\partial/\partial t = 0$ and so the terms under the x -derivatives in the equations (3.15)-(3.17) are constant, i.e.

$$\rho v_x = \text{const} = J, \quad (3.18)$$

$$\rho v_x^2 + p = \text{const} = I, \quad (3.19)$$

$$(\rho v_x) (\frac{1}{2}v_x^2 + e + p/\rho) = \text{const} = J E, \quad (3.20)$$

We assume a perfect gas with

$$p = (\gamma - 1)\rho e \quad (3.21)$$

and substitute

$$e = \frac{1}{\gamma - 1} \frac{p}{\rho}, \quad \text{and} \quad e + \frac{p}{\rho} = \frac{\gamma}{\gamma - 1} \frac{p}{\rho}. \quad (3.22)$$

This allows us rewrite the quantities that are constant across the shock in the form

$$\rho v_x = J, \quad (3.23)$$

$$\rho v_x^2 + p = I, \quad (3.24)$$

$$\frac{1}{2}v_x^2 + \frac{\gamma}{\gamma-1} \frac{p}{\rho} = E. \quad (3.25)$$

We now eliminate ρ and p to derive a quadratic equation for the velocity. Dividing (3.24) by ρ gives

$$v_x^2 + \frac{p}{\rho} = \frac{I}{\rho}. \quad (3.26)$$

We now eliminate p/ρ using (3.25), so we have

$$v_x^2 + \frac{\gamma-1}{\gamma} \left(E - \frac{1}{2}v_x^2 \right) = \frac{I}{\rho}. \quad (3.27)$$

Finally, we get rid of ρ using (3.23), which gives $\rho^{-1} = v_x/J$, so we have

$$v_x^2 + \frac{\gamma-1}{\gamma} \left(E - \frac{1}{2}v_x^2 \right) - \frac{I}{J}v_x = 0, \quad (3.28)$$

which is a quadratic equation for v_x , which we write as

$$v_x^2 \left(1 - \frac{\gamma-1}{2\gamma} \right) - \frac{I}{J}v_x + \frac{\gamma-1}{\gamma} E = 0. \quad (3.29)$$

Since

$$1 - \frac{\gamma-1}{2\gamma} = \frac{\gamma+1}{2\gamma} \quad (3.30)$$

we have, after multiplying by $2\gamma/(\gamma+1)$,

$$v_x^2 - \frac{2\gamma}{\gamma+1} \frac{I}{J}v_x + 2\frac{\gamma-1}{\gamma+1} E = 0. \quad (3.31)$$

Remembering that the quadratic equation can be written in the form $(v - v_1)(v - v_2) = v^2 - (v_1 + v_2)v + v_1v_2 = 0$, where v_1 and v_2 are the two solutions, we get an expression for $v_1 + v_2$,

$$v_1 + v_2 = \frac{2\gamma}{\gamma+1} \frac{I}{J} \quad (3.32)$$

Note also that

$$\frac{I}{Jv_x} = 1 + \frac{1}{\gamma \text{Ma}^2}, \quad (3.33)$$

where $\text{Ma} = v/c$ is the Mach number and $c^2 = \gamma p/\rho$ is the square of the adiabatic sound speed. We now assume that $\text{Ma} \rightarrow \infty$ on the upstream side of the shock. This means that

$$\frac{I}{Jv_1} = 1, \quad (3.34)$$

or $v_1 = I/J$. Substituting this for I/J in (3.32) we have

$$v_1 + v_2 = \frac{2\gamma}{\gamma + 1} v_1, \quad (3.35)$$

or

$$v_2 = \left(\frac{2\gamma}{\gamma + 1} - 1 \right) v_1 = \frac{2\gamma - (\gamma + 1)}{\gamma + 1} v_1 = -\frac{\gamma - 1}{\gamma + 1} v_1. \quad (3.36)$$

The mass ratio on the two sides of the shock is then by (3.23) given by

$$\frac{\rho_2}{\rho_1} = \frac{v_1}{v_2} = \frac{\gamma + 1}{\gamma - 1}. \quad (3.37)$$

For $\gamma = 5/3$ we have

$$\frac{\rho_2}{\rho_1} = \frac{5 + 3}{5 - 3} = \frac{8}{2} = 4. \quad (3.38)$$

For smaller values of γ , for example $\gamma = 7/5$, we have

$$\frac{\rho_2}{\rho_1} = \frac{7 + 5}{7 - 5} = \frac{12}{2} = 6. \quad (3.39)$$

Generalisation to arbitrary shock strength

The assumption of arbitrarily strong shocks was not necessary. All we need to know is all the quantities on one side of the shock. We use eq. (3.33) as it is (without assuming $\text{Ma} \rightarrow \infty$) in eq. (3.32) and obtain

$$v_1 + v_2 = \frac{2\gamma}{\gamma + 1} \left(1 + \frac{1}{\gamma \text{Ma}^2} \right) v_1, \quad (3.40)$$

so

$$\frac{v_2}{v_1} = \frac{2\gamma}{\gamma + 1} \left(1 + \frac{p_1}{\rho_1 v_1^2} \right) - 1 \quad (3.41)$$

Example: suppose $v_1 = 5$, $\rho_1 = p_1 = 1$ and $\gamma = 5/3$, then

$$\frac{v_2}{v_1} = \frac{2 \times 5}{5 + 3} \left(1 + \frac{1}{25} \right) - 1 = \frac{5}{4} \frac{26}{25} - 1 = \frac{1}{2} \frac{13}{5} - 1 = \frac{13}{10} - 1 = 0.3, \quad (3.42)$$

so $v_2 = 5 \times 0.3 = 1.5 = 3/2$. From continuity we have $\rho_2 = \rho_1 v_1 / v_2 = 5/1.5 = 10/3 \approx 3.33$. Finally, for the pressure we

$$\rho_1 v_1^2 + p_1 = \rho_2 v_2^2 + p_2, \quad (3.43)$$

so

$$p_2 = p_1 + \rho_1 v_1^2 - \rho_2 v_2^2 = 1 + 25 - \frac{10}{3} \left(\frac{3}{2} \right)^2 = 26 - \frac{15}{2} = \frac{52 - 15}{2} = \frac{37}{2} = 18.5 \quad (3.44)$$

With this we can now also work out the entropies on the 2 sides of the shock. In the upstream (unshocked) part we have for the entropy in units of c_p ,

$$s_1/c_p = \frac{1}{\gamma} \ln p_1 - \ln \rho_1 = 0, \quad (3.45)$$

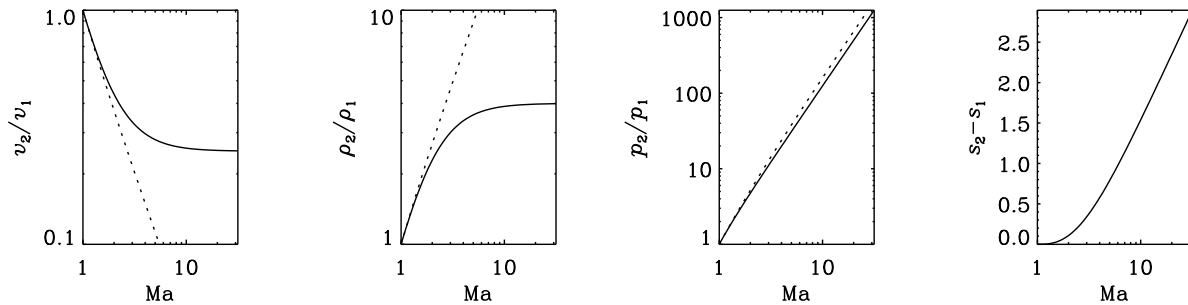


Figure 3.1: Relative velocity, density, pressure and entropy behind the shock. The dotted line refers to the polytropic case where the entropy cannot increase behind the shock.

because $p_1 = \rho_1 = 1$ and $\ln 1 = 0$. On the downstream side the shocked material has $s_2 = 0.55$.

The results are plotted in figure 3.1 for different values of the Mach number.

For polytropic equations of state, with $p = K\rho^\Gamma$, the energy equation is no longer used, so there are only the following two conserved quantities,

$$J = \rho v, \quad I = \rho v^2 + K\rho^\gamma. \quad (3.46)$$

The dependence of velocity, density, pressure, and entropy jumps on the upstream Mach number is plotted in figure ?? for the case $\gamma = 5/3$ and compared with the polytropic case using $\Gamma = \gamma$. Note that in this case the density jump can be arbitrarily strong and is no longer limited by the ratio 4 found for $\gamma = 5/3$. This is however really just an artefact.

Chapter 4

Dimensional Analysis

Dimensional analysis can sometimes be a useful means for understanding the underlying physics. The main difficulties with this approach are twofold. Firstly, it is not always easy to identify the few quantities that govern some physical process. Secondly, there could be some *nondimensional* parameter that may not be close to unity. In that case one gets the right scaling, but quantitatively the result may be off by a few orders of magnitude. In the following we consider a few examples where dimensional analysis works alright: blast waves and various types of turbulence.

4.1 Blast waves

Blast waves are shocks that result from a point-like explosion. Those shocks typically form an expanding spherical shell. In astrophysics such blast waves result from supernova explosions, for example. However, a great deal of theoretical progress has been made in the 40ties when people tried to understand nuclear bombs.

The idea is this. Say we want to calculate the radius of the shell R at any given time t , so we want to know $R = R(t)$. We assume that the solution depends on the total energy E that is liberated during the explosion and on the density ρ of the ambient medium. Those few assumptions alone suffice to generate a nontrivial expression for the solution $R = R(t)$. We make the assumption of a powerlaw,

$$R(t) = \alpha E^a \rho^b t^c, \quad (4.1)$$

where α is a nondimensional constant that we suppose to be of order unity. The values of a , b , and c are solely determined by matching the dimensions on both sides of this equation. The dimensions (indicated by square brackets) of the various quantities are

$$[R] = \text{m}, \quad (4.2)$$

$$[E] = \text{kg m}^2 \text{s}^{-2} \quad (4.3)$$

$$[\rho] = \text{kg m}^{-3} \quad (4.4)$$

$$[t] = \text{s}^{-1} \quad (4.5)$$

Thus, the dimensions on both sides of eq. (4.17) are

$$\text{m}^1 = \text{kg}^a \text{m}^{2a} \text{s}^{-2a} \text{kg}^b \text{m}^{-3b} \text{s}^c. \quad (4.6)$$

To match the dimensions on both sides of the equation we have

$$\text{for the dimension of m: } 1 = 2a - 3b, \quad (4.7)$$

$$\text{for the dimension of s: } 0 = -2a + c. \quad (4.8)$$

$$\text{for the dimension of kg: } 0 = a + b. \quad (4.9)$$

Thus, $b = -a$ and $1 = 2a + 3b = 5a$, so $a = 1/5$ and therefore $b = -1/5$ and $c = 2a = 2/5$. Thus the final expression for $R(t)$ takes the form

$$R(t) = \alpha E^{1/5} \rho^{-1/5} t^{2/5}. \quad (4.10)$$

We can check this formula by plugging in values for the supernova explosion in the Crab nebula in the year 1054. We adopt the typical value of $E = 10^{51}$ erg which, because it is in cgs units, has to be converted into SI units. Let's do that now:

$$10^{51} \text{ erg} = 10^{51} \text{ g} \left(\frac{\text{cm}}{\text{s}} \right)^2 = 10^{51} \times 10^{-3} \text{ kg} \left(\frac{10^{-2} \text{ m}}{\text{s}} \right)^2 = 10^{51} \times 10^{-7} \text{ kg m}^2 \text{s}^{-2} = 10^{44} \text{ kg m}^2 \text{s}^{-2}. \quad (4.11)$$

We may assume $\rho = 10^{-21} \text{ kg/m}^3$. For $t = 10^3 \text{ yr}$ this leads to

$$R \approx 10^{44/5} 10^{21/5} (3 \times 10^{10} \text{ s})^{2/5} \text{ m} \approx 10^{85/5} \text{ m} = 10^{17} \text{ m} = 3 \text{ pc} \quad (4.12)$$

(Remember that $1 \text{ pc} \approx 3 \times 10^{16} \text{ m}$.) This is in excellent agreement with the observed radius of the ejecta which is now roughly 3 pc .¹

4.2 Blast waves in 2-D

The dimension of ρ is kg m^{-3} , so therefore the dimension of $\Sigma = \int \rho dz$ is kg m^{-2} . Assume that the radius $R(t)$ of the blast wave is governed by the energy, E , deliberated during the explosion and the surface density, Σ , of the surrounding medium. Let's make a power law ansatz of the form

$$R(t) = \alpha E^a \Sigma^b t^c, \quad (4.13)$$

¹For numerical adaptive mesh calculations of supernova remnants (SNRs) see Dorfi (1998, in LeVeque et al., Computational Astrophysics, Springer). For applications of SNRs to driving interstellar turbulence see M. Korpi et al. (1999, ApJ 514, L99). A very nice calculation of the relativistic blast wave problem, which is relevant to the fireball in gamma ray bursters, see S. Kobayashi, T. Piran, & R. Sari (1999, ApJ 513, 669)

where α is a dimensionless coefficient. In order that this equation is valid dimensionally, we have to require that the dimensions on both sides are the same. The dimensions of the various quantities are

$$[R] = \text{m}, \quad [E] = \text{kg m}^2 \text{s}^{-2}, \quad [\Sigma] = \text{kg m}^{-2}, \quad [t] = \text{s}. \quad (4.14)$$

This yields on the two sides of (4.13)

$$\text{m}^1 = \text{kg}^a \text{m}^{2a} \text{s}^{-2a} \text{kg}^b \text{m}^{-2b} \text{s}^c. \quad (4.15)$$

To balance the power of cm on the sides we have to require $1 = 2a - 2b$. Furthermore, to get the power of g right we have to require that $0 = a + b$. Finally, to get the dimension of s right we have $0 = -2a + c$. This is a system of 3 linear equations for a , b and c . From the first two equations we have $b = -a$, so $1 = 2a + 2a = 4a$, ie $a = 1/4$. Therefore, $b = -1/4$ and $c = 1/2$, so (4.13) takes the form

$$R(t) = \alpha E^{1/4} \Sigma^{-1/4} t^{1/2}. \quad (4.16)$$

Such blast waves could play a role if the flow is constrained for example by rotation or by magnetic fields to be two-dimensional. In principle such an analysis could also be done for one-dimensional blasts.

4.3 Three-dimensional turbulence

In three-dimensional turbulence the flux of energy, ϵ , is constant and independent of wavenumber k . Using dimensional arguments one can find the energy spectrum $E(k)$ by noting that the energy spectrum is normalised such that $\int_0^\infty E(k) dk = \frac{1}{2} \langle \mathbf{u}^2 \rangle$ and that $E(k)$ depends only ϵ and k . Thus, we write

$$E(k) = C \epsilon^a k^b, \quad (4.17)$$

where C is a dimensionless constant, and a and b are exponents that are determined by matching the dimensions on both sides of the equation. The dimensions (indicated by square brackets) of $E(k)$, ϵ , and k are

$$[E(k)] = \text{m}^3 \text{s}^{-2} \quad (4.18)$$

$$[\epsilon] = \text{m}^2 \text{s}^{-3} \quad (4.19)$$

$$[k] = \text{m}^{-1} \quad (4.20)$$

Thus, the dimensions on both sides of eq. (4.17) are

$$\text{m}^3 \text{s}^{-2} = \text{m}^{2a} \text{s}^{-3a} \text{m}^{-b}. \quad (4.21)$$

To match the dimensions on both sides of the equation we have

$$\text{for the dimension of m: } 3 = 2a - b, \quad (4.22)$$

$$\text{for the dimension of s: } -2 = -3a. \quad (4.23)$$

Thus, $a = 2/3$ and $b = 2a - 3 = 4/3 - 3 = -5/3$, so the spectrum is given by

$$E(k) = C \epsilon^{2/3} k^{-5/3}, \quad (4.24)$$

4.4 Two-dimensional turbulence

In two-dimensional turbulence the rate of enstrophy injection, $\beta = \frac{d}{dt} \langle \omega^2 \rangle$, is constant and independent of wavenumber k . Using dimensional arguments one can find the energy spectrum $E(k)$ by noting that the energy spectrum is normalised such that $\int_0^\infty E(k) dk = \frac{1}{2} \langle \mathbf{u}^2 \rangle$ and that $E(k)$ depends only β and k . Thus, we write

$$E(k) = C \beta^a k^b, \quad (4.25)$$

where C is a dimensionless constant, and a and b are exponents that are determined by matching the dimensions on both sides of the equation. The dimensions (indicated by square brackets) of $E(k)$, β , and k are

$$[E(k)] = \text{m}^3 \text{s}^{-2} \quad (4.26)$$

$$[\beta] = \text{s}^{-3} \quad (4.27)$$

$$[k] = \text{m}^{-1} \quad (4.28)$$

Thus, the dimensions on both sides of eq. (4.25) are

$$\text{m}^3 \text{s}^{-2} = \text{s}^{-3a} \text{m}^{-b}. \quad (4.29)$$

To match the dimensions on both sides of the equation we have

$$\text{for the dimension of m: } 3 = -b, \quad (4.30)$$

$$\text{for the dimension of s: } -2 = -3a. \quad (4.31)$$

Thus, $b = -3$ and $a = 2/3$, so the spectrum is given by

$$E(k) = C \beta^{2/3} k^{-3}, \quad (4.32)$$

4.5 Hydromagnetic turbulence

Using dimensional arguments one can find the energy spectrum $E(k)$ for hydromagnetic turbulence, by assuming that the energy transfer is governed by the geometric mean of the Alfvén speed and the energy transfer rate ϵ . Thus, we assume the spectrum to be of the form

$$E(k) = C (v_A \epsilon)^a k^b, \quad (4.33)$$

where C is a dimensionless constant, v_A is the Alfvén speed, ϵ (with dimension $\text{m}^2 \text{s}^{-3}$) the energy injection rate, and k the wavenumber. The dimensions (indicated by square brackets) of $E(k)$, v_A , ϵ , and k are

$$[E(k)] = \text{m}^3 \text{s}^{-2} \quad (4.34)$$

$$[v_A] = \text{m} \text{s}^{-1} \quad (4.35)$$

$$[\epsilon] = \text{m}^2 \text{s}^{-3} \quad (4.36)$$

$$[k] = \text{m}^{-1} \quad (4.37)$$

Thus, the dimensions on both sides of eq. (4.33) are

$$\text{m}^3 \text{s}^{-2} = \text{m}^a \text{s}^{-a} \text{m}^{2a} \text{s}^{-3a} \text{m}^{-b}. \quad (4.38)$$

To match the dimensions on both sides of the equation we have

$$\text{for the dimension of m: } 3 = 3a - b, \quad (4.39)$$

$$\text{for the dimension of s: } -2 = -4a. \quad (4.40)$$

Thus, $a = 1/2$ and so $b = 3a - 3 = 3/2 - 3 = -3/2$, so the spectrum is given by

$$E(k) = C (v_A \epsilon)^{1/2} k^{-3/2}, \quad (4.41)$$

A final comment is here in order. Although the $k^{-3/2}$ scaling has been verified in two-dimensional numerical simulations, there is now mounting evidence in favour of the more conventional $k^{-5/3}$ scaling even in the magnetic case.²

²See recent paper by Biskamp & Müller 1999, Phys. Plasmas **7**, 4889

Chapter 5

Magnetic Fields

The evolution of the magnetic field is governed by Maxwell's equations and by “material” equations, such as Ohm's law. In Maxwell's equations the Faraday displacement current is neglected, which is valid if either the relevant velocities in the problem are nonrelativistic (much smaller than the speed of light), or if the conductivity of the medium is so poor that there are no (ordinary) currents and only the displacement current balancing the $\nabla \times \mathbf{B}$ term in Faraday's equation. The latter leads to radio waves, for example. Both limiting cases are not considered in this course, although they may be of interest in some extreme astrophysical cases (near black holes or in the big bang).

5.1 The induction equation

The magnetic field, $\mathbf{B}$, is governed by the equations

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0, \quad (5.1)$$

$$\mathbf{J} = \nabla \times \mathbf{B} / \mu_0, \quad (5.2)$$

$$\mathbf{E} = -\mathbf{v} \times \mathbf{B}, \quad (5.3)$$

where $\mathbf{E}$ is the electric field, $\mathbf{J}$ the current density, $\mathbf{v}$ the velocity, and μ_0 the permeability. The field always satisfies $\nabla \cdot \mathbf{B}$, but if this is satisfied by the initial condition, the evolution equation (5.1) will satisfy this automatically, because the divergence of a curl vanishes.

Substituting now $\mathbf{E}$ from eq. (5.3) into (5.1) yields the induction equation

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}). \quad (5.4)$$

In many cases there is also resistivity, which can be important, but for now this will be ignored. However it is sometimes useful to rewrite the induction term using the identity

$$\nabla \times (\mathbf{v} \times \mathbf{B}) = -\mathbf{v} \cdot \nabla \mathbf{B} + \mathbf{B} \cdot \nabla \mathbf{v} + \mathbf{v} \nabla \cdot \mathbf{B} - \mathbf{B} \nabla \cdot \mathbf{v}. \quad (5.5)$$

Here, the term $\mathbf{v} \nabla \cdot \mathbf{B}$ vanishes, because $\nabla \cdot \mathbf{B}$ vanishes. Thus, the induction equation can be written in the form

$$\frac{\partial \mathbf{B}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{B} = \mathbf{B} \cdot \nabla \mathbf{v} - \mathbf{B} \nabla \cdot \mathbf{v}. \quad (5.6)$$

5.2 The momentum equation

The momentum equation is supplemented by an extra term, the Lorentz force $\mathbf{J} \times \mathbf{B}$, which then takes the form

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{1}{\rho} \nabla p + \frac{1}{\rho} \mathbf{J} \times \mathbf{B} + \dots, \quad (5.7)$$

where the dots refer to other possible terms (friction, Coriolis force, etc.) The Lorentz force can also be written in the form

$$\mathbf{J} \times \mathbf{B} = \frac{1}{\mu_0} (\mathbf{B} \cdot \nabla) \mathbf{B} - \nabla \left(\frac{\mathbf{B}^2}{2\mu_0} \right). \quad (5.8)$$

The first term is commonly called the tension force, because it gives the derivative of the magnetic field in the direction of the magnetic field.

5.3 The magnetic energy equation

By dotting eq. (5.1) with $\mu_0^{-1} \mathbf{B}$ we obtain

$$\frac{\partial}{\partial t} \left(\frac{\mathbf{B}^2}{2\mu_0} \right) + \frac{1}{\mu_0} \mathbf{B} \cdot \nabla \times \mathbf{E} = 0. \quad (5.9)$$

Noting that $\nabla \cdot (\mathbf{E} \times \mathbf{B}) = \epsilon_{ijk} \partial_i (E_j B_k)$ we can write, using the product rule,

$$\nabla \cdot (\mathbf{E} \times \mathbf{B}) = \mathbf{B} \cdot \nabla \times \mathbf{E} - \mathbf{E} \cdot \nabla \times \mathbf{B}. \quad (5.10)$$

[Remember that $\epsilon_{ijk} = \epsilon_{jki} = -\epsilon_{jik}$.] Using this relation we find that

$$\frac{\partial}{\partial t} \left(\frac{\mathbf{B}^2}{2\mu_0} \right) + \nabla \cdot \left(\frac{\mathbf{E} \times \mathbf{B}}{\mu_0} \right) + \mathbf{J} \cdot \mathbf{E} = 0. \quad (5.11)$$

Writing $\mathbf{J} \cdot \mathbf{E} = \mathbf{J} \cdot \mathbf{v} \times \mathbf{B} = \epsilon_{ijk} J_i v_j B_k$ we obtain the energy equation in the form

$$\frac{\partial}{\partial t} \left(\frac{\mathbf{B}^2}{2\mu_0} \right) + \nabla \cdot \left(\frac{\mathbf{E} \times \mathbf{B}}{\mu_0} \right) + \mathbf{v} \cdot (\mathbf{J} \times \mathbf{B}) = 0, \quad (5.12)$$

or, as an evolution equation for the magnetic energy density, $\mathbf{B}^2/2\mu_0$, we have

$$\frac{\partial}{\partial t} \left(\frac{\mathbf{B}^2}{2\mu_0} \right) = -\nabla \cdot \left(\frac{\mathbf{E} \times \mathbf{B}}{\mu_0} \right) - \mathbf{v} \cdot (\mathbf{J} \times \mathbf{B}). \quad (5.13)$$

The last term is important, because it give the work done against the Lorentz force. If we write down the corresponding equation for the kinetic energy, that is obtained by dotting the momentum equation

with $\rho \mathbf{v}$, we obtain the term $+\mathbf{v} \cdot (\mathbf{J} \times \mathbf{B})$ on the rhs of that equation. In the equation for the sum of kinetic and magnetic energies this term then cancels. That means that this term describes the transfer of energy between kinetic and magnetic energy reservoirs. In many cases $\mathbf{v} \cdot (\mathbf{J} \times \mathbf{B})$ is negative, i.e. work is done against the Lorentz force and can lead to an increase of magnetic energy. This is the case in astrophysical dynamos. Again a hot research topic.

5.4 Hydromagnetic equations in conservative form

It is possible to write the magnetohydrodynamic (MHD) equations in conservative form, i.e. in the form

$$\frac{\partial}{\partial t}(\text{something}) + \nabla \cdot (\text{some corresponding flux}) = 0. \quad (5.14)$$

In the case of the MHD equations the momentum and energy equations can be written in the form

$$\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_j} \left[\rho v_i v_j + \delta_{ij} p + \frac{1}{\mu_0} (B_i B_j - \frac{1}{2} \delta_{ij} \mathbf{B}^2) \right] = 0, \quad (5.15)$$

$$\frac{\partial}{\partial t} (\frac{1}{2} \rho \mathbf{v}^2 + \rho e + \frac{1}{2\mu_0} \mathbf{B}^2) + \nabla \cdot \left[\mathbf{v} \left(\frac{1}{2} \rho \mathbf{v}^2 + \rho e + p \right) + \mathbf{E} \times \mathbf{B} \right] = 0. \quad (5.16)$$

Here we have used the continuity, momentum and energy equations together with the induction equation. The induction equation, $\partial \mathbf{B} / \partial t = \nabla \times (\mathbf{u} \times \mathbf{B})$, is readily written in conservative by just expanding the coefficients, so

$$\frac{\partial B_i}{\partial t} + \frac{\partial}{\partial x_j} (u_j B_i - u_i B_j) = 0, \quad (5.17)$$

where the magnetic diffusivity is ignored.

In a sense those equations in conservative form are more fundamental than the momentum and energy equations that we have used so far, because if there are extra terms, for example in the continuity equation due to chemical reactions for example, this may yield additional terms in the momentum equation, but the equations in conservative form remain unchanged, unless those reactions lead to momentum injection.

5.5 One-dimensional MHD shocks

In one dimension the conservation equations can be integrated provided we are in the frame of reference comoving with the shock. We assume $B_x = \text{const}$ and $B_z = 0$, so we have

$$\rho v_x = J \quad (5.18)$$

$$\rho v_x^2 + p + \frac{1}{2\mu_0} (B_x^2 - B_y^2) = I \quad (5.19)$$

$$\frac{1}{2} v_x^2 + \frac{\gamma}{\gamma - 1} \frac{p}{\rho} = E \quad (5.20)$$

$$v_x B_y = M \tag{5.21}$$

where J , I , E , and M are integration constants. From this one can derive jump conditions very similar to those without fields. One ends up with a cubic equation (or a quartic equation in the more general case where a velocity component in the direction of the shock is allowed for). The different solutions correspond to different MHD wave types (slow and fast magnetosonic and Alfvén), but not always all those different types are possible, which is when the upstream velocity would be smaller than any of the other wave speeds. In the following we discuss those various wave types, although we won't go into this very interesting field of MHD shocks to any greater detail.

Chapter 6

Alfvén and magnetosonic waves

In this chapter we demonstrate the technique of linearising the MHD equations, which can then be used to study phenomena of small amplitude. This applies mainly to waves (provided they are of small amplitude), but it also applies to weak perturbations that can in some systems grow exponentially and would eventually no longer be small. In that case such an analysis can establish the possibility of an instability.

6.1 Linearising the MHD equations

We now consider the isothermal MHD equations in the form

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho \mathbf{v} \cdot \nabla \mathbf{v} = -c_s^2 \nabla \rho - \nabla \left(\frac{\mathbf{B}^2}{2\mu_0} \right) + \frac{1}{\mu_0} (\mathbf{B} \cdot \nabla) \mathbf{B}. \quad (6.1)$$

Whenever we linearise, we have to linearise about something, which must be a solution to the equations. Those solutions may well be complicated, but then we won't be able to do the analysis so easily on the blackboard. Therefore we now go for simple solutions. One solution is where the magnetic field is constant and uniform, i.e. $\mathbf{B} = \mathbf{B}_0$, and where the density is also constant, i.e. $\rho = \rho_0$, but the velocity is zero, i.e. $\mathbf{v} = \mathbf{v}_0 = 0$. In that case the terms on the lhs of eq. (6.1) vanish. The terms on the rhs of eq. (6.1) also vanish because ρ_0 and $\mathbf{B}_0$ are uniform. So we do have a solution now.

We now proceed by linearising term by term, i.e. we write

$$\rho = \rho_0 + \rho', \quad (6.2)$$

$$\mathbf{v} = \mathbf{v}_0 + \mathbf{v}' = \mathbf{v}', \quad (6.3)$$

$$\mathbf{B} = \mathbf{B}_0 + \mathbf{B}'. \quad (6.4)$$

The first term in (6.1) then becomes

$$\rho \frac{\partial \mathbf{v}}{\partial t} = (\rho_0 + \rho') \frac{\partial \mathbf{v}'}{\partial t} = \rho_0 \frac{\partial \mathbf{v}'}{\partial t} + \rho' \frac{\partial \mathbf{v}'}{\partial t}. \quad (6.5)$$

Here the second term is quadratic in the perturbations. Since those perturbations are small, then something small squared will be even smaller and will hence be neglected. The advection term is already nonlinear in the perturbations, because there is no zero-order term, so it vanishes altogether. Next, the pressure gradient term is linear already, so we have

$$c_s^2 \nabla \rho = c_s^2 \nabla \rho', \quad (6.6)$$

because ρ_0 is constant. For the magnetic pressure gradient we have

$$\nabla \left(\frac{\mathbf{B}^2}{2\mu_0} \right) = \nabla \left(\frac{\mathbf{B}_0 \cdot \mathbf{B}'}{\mu_0} \right) + \text{quadratic terms}, \quad (6.7)$$

because

$$\mathbf{B}^2 = (\mathbf{B}_0 + \mathbf{B}') \cdot (\mathbf{B}_0 + \mathbf{B}') = \mathbf{B}_0^2 + 2\mathbf{B}_0 \cdot \mathbf{B}' + \mathbf{B}'^2, \quad (6.8)$$

of which only the second term survives. The first one is constant and gives no contribution under the gradient, and the last term is quadratic in the perturbations. Finally, the magnetic stretching term gives simply

$$\frac{1}{\mu_0} (\mathbf{B} \cdot \nabla) \mathbf{B} = \frac{1}{\mu_0} (\mathbf{B}_0 \cdot \nabla) \mathbf{B}' + \text{quadratic terms}. \quad (6.9)$$

Thus, our linearised momentum equation takes the form

$$\rho_0 \frac{\partial \mathbf{v}'}{\partial t} = -c_s^2 \nabla \rho' - \nabla \left(\frac{\mathbf{B}_0 \cdot \mathbf{B}'}{\mu_0} \right) + \frac{1}{\mu_0} (\mathbf{B}_0 \cdot \nabla) \mathbf{B}'. \quad (6.10)$$

You can imagine that with a little bit of experience one can write down those equations straight away. We do this now with the remaining equations, the induction and continuity equations, looking at equations (5.6) and (1.1),

$$\frac{\partial \mathbf{B}'}{\partial t} = \mathbf{B}_0 \cdot \nabla \mathbf{v}' - \mathbf{B}_0 \nabla \cdot \mathbf{v}'. \quad (6.11)$$

$$\frac{\partial \rho'}{\partial t} + \rho_0 \nabla \cdot \mathbf{v}' = 0, \quad (6.12)$$

Next, we go through some examples of various degrees of interest.

6.2 Alfvén waves in the presence of a vertical magnetic field

We assume that all variables can be decomposed into plane waves in the form

$$v'_x = \hat{v}_x e^{i\mathbf{k} \cdot \mathbf{x} - i\omega t}, \quad (6.13)$$

$$v'_y = \hat{v}_y e^{i\mathbf{k} \cdot \mathbf{x} - i\omega t}, \quad (6.14)$$

$$v'_z = \hat{v}_z e^{i\mathbf{k} \cdot \mathbf{x} - i\omega t}, \quad (6.15)$$

$$B'_x = \hat{B}_x e^{i\mathbf{k} \cdot \mathbf{x} - i\omega t}, \quad (6.16)$$

$$B'_y = \hat{B}_y e^{i\mathbf{k} \cdot \mathbf{x} - i\omega t}, \quad (6.17)$$

$$B'_z = \hat{B}_z e^{i\mathbf{k} \cdot \mathbf{x} - i\omega t}, \quad (6.18)$$

$$\rho' = \hat{\rho} e^{i\mathbf{k} \cdot \mathbf{x} - i\omega t}, \quad (6.19)$$

which allows us to derive a set of algebraic equations. For the velocity components we have

$$-i\omega \hat{v}_x = -ik_x \hat{\rho} \frac{c_s^2}{\rho_0} - ik_x \hat{B}_z \frac{B_0}{\mu_0 \rho_0} + ik_z \hat{B}_x \frac{B_0}{\mu_0 \rho_0} \quad (6.20)$$

$$-i\omega \hat{v}_y = -ik_y \hat{\rho} \frac{c_s^2}{\rho_0} - ik_y \hat{B}_z \frac{B_0}{\mu_0 \rho_0} + ik_z \hat{B}_y \frac{B_0}{\mu_0 \rho_0} \quad (6.21)$$

$$-i\omega \hat{v}_z = -ik_z \hat{\rho} \frac{c_s^2}{\rho_0} \quad (6.22)$$

where the last two magnetic terms have cancelled. The continuity equation is

$$-i\omega \hat{\rho} = -ik_x \rho_0 \hat{u}_x - ik_y \rho_0 \hat{u}_y - ik_z \rho_0 \hat{u}_z \quad (6.23)$$

The magnetic equations are now for the form

$$-i\omega \hat{B}_x = +ik_z B_0 \hat{u}_x \quad (6.24)$$

$$-i\omega \hat{B}_y = +ik_z B_0 \hat{u}_y \quad (6.25)$$

$$-i\omega \hat{B}_z = -ik_x B_0 \hat{u}_x - ik_y B_0 \hat{u}_y \quad (6.26)$$

where, again, the ik_z terms have cancelled in the last equation.

It is now convenient to write those equations in matrix form

$$\begin{pmatrix} -i\omega & 0 & 0 & ik_x \frac{c_s^2}{\rho_0} & -ik_z \frac{B_0}{\mu_0 \rho_0} & 0 & +ik_x \frac{B_0}{\mu_0 \rho_0} \\ 0 & -i\omega & 0 & ik_y \frac{c_s^2}{\rho_0} & 0 & -ik_z \frac{B_0}{\mu_0 \rho_0} & +ik_y \frac{B_0}{\mu_0 \rho_0} \\ 0 & 0 & -i\omega & ik_z \frac{c_s^2}{\rho_0} & 0 & 0 & 0 \\ ik_x \rho_0 & ik_y \rho_0 & ik_z \rho_0 & -i\omega & 0 & 0 & 0 \\ -ik_z B_0 & 0 & 0 & 0 & -i\omega & 0 & 0 \\ 0 & -ik_z B_0 & 0 & 0 & 0 & -i\omega & 0 \\ ik_x B_0 & ik_y B_0 & 0 & 0 & 0 & 0 & -i\omega \end{pmatrix} \begin{pmatrix} \hat{v}_x \\ \hat{v}_y \\ \hat{v}_z \\ \hat{\rho} \\ \hat{B}_x \\ \hat{B}_y \\ \hat{B}_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad (6.27)$$

We won't consider this rather large system of equations at this point. Instead we want to observe what happens when we assume that the system is one-dimensional. Hence, we ignore the x and y

components of the wave vector, so our matrix becomes

$$\begin{pmatrix} -i\omega & 0 & 0 & 0 & -ik_z \frac{B_0}{\mu_0 \rho_0} & 0 & 0 \\ 0 & -i\omega & 0 & 0 & 0 & -ik_z \frac{B_0}{\mu_0 \rho_0} & 0 \\ 0 & 0 & -i\omega & ik_z \frac{c_s^2}{\rho_0} & 0 & 0 & 0 \\ 0 & 0 & ik_z \rho_0 & -i\omega & 0 & 0 & 0 \\ -ik_z B_0 & 0 & 0 & 0 & -i\omega & 0 & 0 \\ 0 & -ik_z B_0 & 0 & 0 & 0 & -i\omega & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -i\omega \end{pmatrix} \begin{pmatrix} \hat{v}_x \\ \hat{v}_y \\ \hat{v}_z \\ \hat{\rho} \\ \hat{B}_x \\ \hat{B}_y \\ \hat{B}_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad (6.28)$$

We notice that the last equation disappears altogether. We notice further that the third and fourth equations decouple from the remaining four equations. In order to have non-trivial solutions (where the hatted variables themselves don't vanish), we have to require that the determinant of the matrix vanishes. Because the matrix decouples into two matrices, M_1 and M_2 , we have to require that either $\det M_1 = 0$ or $\det M_2 = 0$, where

$$M_1 = \begin{pmatrix} -i\omega & 0 & -ik_z \frac{B_0}{\mu_0 \rho_0} & 0 \\ 0 & -i\omega & 0 & -ik_z \frac{B_0}{\mu_0 \rho_0} \\ -ik_z B_0 & 0 & -i\omega & 0 \\ 0 & -ik_z B_0 & 0 & -i\omega \end{pmatrix}, \quad M_2 = \begin{pmatrix} -i\omega & ik_z \frac{c_s^2}{\rho_0} \\ ik_z \rho_0 & -i\omega \end{pmatrix} \quad (6.29)$$

There are two separate dispersion relations. From $\det M_2 = 0$ we have

$$\omega^2 = c_s^2 k_z^2. \quad (6.30)$$

The resulting dispersion relation from $\det M_1 = 0$ is

$$\omega^4 - 2v_A^2 k^2 \omega^2 + v_A^4 k^4 = 0. \quad (6.31)$$

This is a biquadratic equation, or a quadratic equation in ω^2 . However, in this case it can also be written as

$$(\omega^2 - v_A^2 k^2)^2 = 0, \quad (6.32)$$

which simply means that there are Alfvén waves where either the x or the y components of the field are involved.

6.3 One-dimensional Alfvén waves revisited

Given that in one dimension the Alfvén waves simplify significantly we begin all over again, focusing attention immediately on the essential points. The linearised, pressureless, ideal MHD equations in one dimension, in the presence of a vertical magnetic field, are

$$\frac{\partial v'_x}{\partial t} = \frac{B_0}{\mu_0 \rho_0} \frac{\partial B'_x}{\partial z}, \quad (6.33)$$

$$\frac{\partial B'_x}{\partial t} = B_0 \frac{\partial v'_x}{\partial z}, \quad (6.34)$$

Differentiating the first equation in time and inserting the second equation for $\partial B'_x/\partial t$ yields a wave equation, similar to the equation describing sound waves. In the more complicated situations described below it turns out to be easier however to take the solution to be in the form

$$v'_x = \hat{v}_x e^{ikz - i\omega t}, \quad (6.35)$$

$$B'_x = \hat{B}_x e^{ikz - i\omega t}, \quad (6.36)$$

which allows us to derive a set of two algebraic equations,

$$-i\omega \hat{v}_x - 2\Omega \hat{v}_y = \frac{B_0}{\rho_0 \mu_0} ik \hat{B}_x, \quad (6.37)$$

$$-i\omega \hat{B}_x = B_0 ik \hat{v}_x. \quad (6.38)$$

It is convenient to write those equations in matrix form

$$\begin{pmatrix} -i\omega & -ik \frac{B_0}{\mu_0 \rho_0} \\ -ik B_0 & -i\omega \end{pmatrix} \begin{pmatrix} \hat{v}_x \\ \hat{B}_x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad (6.39)$$

For a nontrivial solution the determinant of the governing matrix has to vanish

$$\det M \equiv \det \begin{pmatrix} -i\omega & -ik \frac{B_0}{\mu_0 \rho_0} \\ -ik B_0 & -i\omega \end{pmatrix} = -\omega^2 + \frac{B_0^2}{\mu_0 \rho_0} k^2 = 0, \quad (6.40)$$

This leads to the dispersion relation $\omega = \omega(k)$ with

$$\omega = \pm v_A k, \quad (6.41)$$

where v_A is the Alfvén speed with $v_A^2 = B_0^2/(\rho_0 \mu_0)$.

Unlike sound waves, where the restoring force is the pressure gradient, for Alfvén waves the restoring force is the magnetic field. Another important difference is the fact that now the field has only components perpendicular to the direction of propagation. Here the direction of propagation is z , but $\mathbf{B}$ has only a nonvanishing component in the x direction. For sound waves, on the other hand, the velocity has only a component in the direction of propagation.

6.3.1 The effects of rotation

We now assume that we are in a rotating system of reference, where the rotation axis is the z axis, which is also the direction of the applied magnetic field, so $\boldsymbol{\Omega} = (0, 0, \Omega)$ and so the Coriolis force (per unit mass) is

$$\mathbf{F}_{\text{Cor}} = -2\boldsymbol{\Omega} \times \mathbf{v} = \begin{pmatrix} 0 \\ 0 \\ -2\Omega \end{pmatrix} \times \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \begin{pmatrix} +2\Omega v_y \\ -2\Omega v_x \\ 0 \end{pmatrix}. \quad (6.42)$$

Adding this term to the equations causes immediately some coupling to the y components of both the velocity and the magnetic field. The linearised, pressureless, ideal MHD equations in one dimension, in the presence of rotation, are then

$$\frac{\partial v'_x}{\partial t} - 2\Omega v'_y = \frac{B_0}{\mu_0 \rho_0} \frac{\partial B'_x}{\partial z}, \quad (6.43)$$

$$\frac{\partial v'_y}{\partial t} + 2\Omega v'_x = \frac{B_0}{\mu_0 \rho_0} \frac{\partial B'_y}{\partial z}, \quad (6.44)$$

$$\frac{\partial B'_x}{\partial t} = B_0 \frac{\partial v'_x}{\partial z}, \quad (6.45)$$

$$\frac{\partial B'_y}{\partial t} = B_0 \frac{\partial v'_y}{\partial z}. \quad (6.46)$$

As usual, we assume the solution to be of the form

$$v'_x = \hat{v}_x e^{ikz - i\omega t}, \quad (6.47)$$

$$v'_y = \hat{v}_y e^{ikz - i\omega t}, \quad (6.48)$$

$$B'_x = \hat{B}_x e^{ikz - i\omega t}, \quad (6.49)$$

$$B'_y = \hat{B}_y e^{ikz - i\omega t}, \quad (6.50)$$

which allows us to derive a set of algebraic equations.

$$-i\omega \hat{v}_x - 2\Omega \hat{v}_y = \frac{B_0}{\rho_0 \mu_0} ik \hat{B}_x, \quad (6.51)$$

$$-i\omega \hat{v}_y + 2\Omega \hat{v}_x = \frac{B_0}{\rho_0 \mu_0} ik \hat{B}_y, \quad (6.52)$$

$$-i\omega \hat{B}_x = B_0 ik \hat{v}_x, \quad (6.53)$$

$$-i\omega \hat{B}_y = B_0 ik \hat{v}_y. \quad (6.54)$$

It is convenient to write those equations in matrix form

$$\begin{pmatrix} -i\omega & -2\Omega & -ik \frac{B_0}{\mu_0 \rho_0} & 0 \\ 2\Omega & -i\omega & 0 & -ik \frac{B_0}{\mu_0 \rho_0} \\ -ik B_0 & 0 & -i\omega & 0 \\ 0 & -ik B_0 & 0 & -i\omega \end{pmatrix} \begin{pmatrix} \hat{v}_x \\ \hat{v}_y \\ \hat{B}_x \\ \hat{B}_y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad (6.55)$$

For a nontrivial solution the determinant of the governing matrix has to vanish

$$\det M \equiv \det \begin{pmatrix} -i\omega & -2\Omega & -ik \frac{B_0}{\mu_0 \rho_0} & 0 \\ 2\Omega & -i\omega & 0 & -ik \frac{B_0}{\mu_0 \rho_0} \\ -ik B_0 & 0 & -i\omega & 0 \\ 0 & -ik B_0 & 0 & -i\omega \end{pmatrix}. \quad (6.56)$$

To calculate the determinant we split it into sub-determinants:

$$\det M = -i\omega \det \begin{pmatrix} -i\omega & 0 & -ik \frac{B_0}{\mu_0 \rho_0} \\ 0 & -i\omega & 0 \\ -ikB_0 & 0 & -i\omega \end{pmatrix} + 2\Omega \det \begin{pmatrix} 2\Omega & 0 & -ik \frac{B_0}{\mu_0 \rho_0} \\ -ikB_0 & -i\omega & 0 \\ 0 & 0 & -i\omega \end{pmatrix} - ik \frac{B_0}{\mu_0 \rho_0} \det \begin{pmatrix} 2\Omega & -i\omega & -ik \frac{B_0}{\mu_0 \rho_0} \\ -ikB_0 & 0 & 0 \\ 0 & -ikB_0 & -i\omega \end{pmatrix}. \quad (6.57)$$

Determinants of 3×3 matrices are easy to calculate, so

$$\det M = -i\omega \left[(-i\omega)^3 - (-i\omega) \left(-k^2 \frac{B_0^2}{\mu_0 \rho_0} \right) \right] + 2\Omega \left[2\Omega(-i\omega)^2 \right] - ik \frac{B_0}{\mu_0 \rho_0} \left[-k^2 \frac{B_0^2}{\mu_0 \rho_0} (-ikB_0) - (-i\omega)^2 (-ikB_0) \right]. \quad (6.58)$$

Thus,

$$\det M = -\omega^2 \left[-\omega^2 + k^2 \frac{B_0^2}{\mu_0 \rho_0} \right] - 4\Omega^2 \omega^2 - k^2 \frac{B_0^2}{\mu_0 \rho_0} \left[-k^2 \frac{B_0^2}{\mu_0 \rho_0} + \omega^2 \right], \quad (6.59)$$

or

$$\det M = \omega^4 - \omega^2 k^2 v_A^2 - 4\Omega^2 \omega^2 + (k^2 v_A^2)^2 - k^2 v_A^2 \omega^2, \quad (6.60)$$

where $v_A^2 = B_0^2/(\mu_0 \rho_0)$ is the square of the Alfvén speed. Setting the determinant to zero leads to the dispersion relation

$$\omega^4 - \omega^2 (2v_A^2 k^2 + 4\Omega^2) + v_A^4 k^4 = 0. \quad (6.61)$$

This is a biquadratic equation, or a quadratic equation in ω^2 .

6.3.2 The effects of rotation and shear

The linearised, pressureless, ideal MHD equations in one dimension, in the presence of rotation and shear with a vertical magnetic field, are

$$\frac{\partial v'_x}{\partial t} - 2\Omega v'_y = \frac{B_0}{\mu_0 \rho_0} \frac{\partial B'_x}{\partial z}, \quad (6.62)$$

$$\frac{\partial v'_y}{\partial t} + \frac{1}{2}\Omega v'_x = \frac{B_0}{\mu_0 \rho_0} \frac{\partial B'_y}{\partial z}, \quad (6.63)$$

$$\frac{\partial B'_x}{\partial t} = B_0 \frac{\partial v'_x}{\partial z}, \quad (6.64)$$

$$\frac{\partial B'_y}{\partial t} = B_0 \frac{\partial v'_y}{\partial z} - \frac{3}{2}\Omega B'_x, \quad (6.65)$$

We assume the solution to be of the form

$$v'_x = \hat{v}_x e^{ikz - i\omega t}, \quad (6.66)$$

$$v'_y = \hat{v}_y e^{ikz - i\omega t}, \quad (6.67)$$

$$B'_x = \hat{B}_x e^{ikz - i\omega t}, \quad (6.68)$$

$$B'_y = \hat{B}_y e^{ikz - i\omega t}, \quad (6.69)$$

which allows us to derive a set of algebraic equations.

$$-i\omega \hat{v}_x - 2\Omega \hat{v}_y = \frac{B_0}{\rho_0 \mu_0} ik \hat{B}_x, \quad (6.70)$$

$$-i\omega \hat{v}_y + \frac{1}{2}\Omega \hat{v}_x = \frac{B_0}{\rho_0 \mu_0} ik \hat{B}_y, \quad (6.71)$$

$$-i\omega \hat{B}_x = B_0 ik \hat{v}_x, \quad (6.72)$$

$$-i\omega \hat{B}_y = B_0 ik \hat{v}_y - \frac{3}{2}\Omega \hat{B}_x. \quad (6.73)$$

It is convenient to write those equations in matrix form

$$\begin{pmatrix} -i\omega & -2\Omega & -ik \frac{B_0}{\mu_0 \rho_0} & 0 \\ \frac{1}{2}\Omega & -i\omega & 0 & -ik \frac{B_0}{\mu_0 \rho_0} \\ -ik B_0 & 0 & -i\omega & 0 \\ 0 & -ik B_0 & \frac{3}{2}\Omega & -i\omega \end{pmatrix} \begin{pmatrix} \hat{v}_x \\ \hat{v}_y \\ \hat{B}_x \\ \hat{B}_y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad (6.74)$$

For a nontrivial solution the determinant of the governing matrix has to vanish

$$\det M \equiv \det \begin{pmatrix} -i\omega & -2\Omega & -ik \frac{B_0}{\mu_0 \rho_0} & 0 \\ \frac{1}{2}\Omega & -i\omega & 0 & -ik \frac{B_0}{\mu_0 \rho_0} \\ -ik B_0 & 0 & -i\omega & 0 \\ 0 & -ik B_0 & \frac{3}{2}\Omega & -i\omega \end{pmatrix}. \quad (6.75)$$

To calculate the determinant we split it into sub-determinants:

$$\begin{aligned} \det M = & -i\omega \det \begin{pmatrix} -i\omega & 0 & -ik \frac{B_0}{\mu_0 \rho_0} \\ 0 & -i\omega & 0 \\ -ik B_0 & \frac{3}{2}\Omega & -i\omega \end{pmatrix} + 2\Omega \det \begin{pmatrix} \frac{1}{2}\Omega & 0 & -ik \frac{B_0}{\mu_0 \rho_0} \\ -ik B_0 & -i\omega & 0 \\ 0 & \frac{3}{2}\Omega & -i\omega \end{pmatrix} \\ & - ik \frac{B_0}{\mu_0 \rho_0} \det \begin{pmatrix} \frac{1}{2}\Omega & -i\omega & -ik \frac{B_0}{\mu_0 \rho_0} \\ -ik B_0 & 0 & 0 \\ 0 & -ik B_0 & -i\omega \end{pmatrix}. \end{aligned} \quad (6.76)$$

Determinants of 3×3 matrices are easy to calculate, so

$$\begin{aligned} \det M = & -i\omega \left[(-i\omega)^3 - (-i\omega) \left(-k^2 \frac{B_0^2}{\mu_0 \rho_0} \right) \right] + 2\Omega \left[\frac{1}{2}\Omega (-i\omega)^2 - k^2 \frac{B_0^2}{\mu_0 \rho_0} \frac{3}{2}\Omega \right] \\ & - ik \frac{B_0}{\mu_0 \rho_0} \left[-k^2 \frac{B_0^2}{\mu_0 \rho_0} (-ik B_0) - (-i\omega)^2 (-ik B_0) \right]. \end{aligned} \quad (6.77)$$

Thus,

$$\det M = -\omega^2 \left[-\omega^2 + k^2 \frac{B_0^2}{\mu_0 \rho_0} \right] + 2\Omega \left[-\frac{1}{2}\Omega\omega^2 - k^2 \frac{B_0^2}{\mu_0 \rho_0} \frac{3}{2}\Omega \right] - k^2 \frac{B_0^2}{\mu_0 \rho_0} \left[-k^2 \frac{B_0^2}{\mu_0 \rho_0} + \omega^2 \right], \quad (6.78)$$

or

$$\det M = \omega^4 - \omega^2 k^2 v_A^2 - \Omega^2 \omega^2 - 3\Omega^2 k^2 v_A^2 + (k^2 v_A^2)^2 - k^2 v_A^2 \omega^2, \quad (6.79)$$

where $v_A^2 = B_0^2/(\mu_0 \rho_0)$ is the square of the Alfvén speed. Setting the determinant to zero leads to the dispersion relation

$$\omega^4 - \omega^2(2v_A^2 k^2 + \Omega^2) + v_A^2 k^2(v_A^2 k^2 - 3\Omega^2) = 0. \quad (6.80)$$

This is a biquadratic equation, or a quadratic equation in ω^2 . There are two solutions for ω^2 , ω_1^2 and ω_2^2 , say. We can therefore write the dispersion relation in the form

$$(\omega^2 - \omega_1^2)(\omega^2 - \omega_2^2) = 0, \quad (6.81)$$

or

$$\omega^4 - \omega^2(\omega_1^2 + \omega_2^2) + \omega_1^2 \omega_2^2 = 0. \quad (6.82)$$

Comparing with eq. (6.80) we see that one of the two solutions is negative, i.e. $\omega_1^2 \omega_2^2 < 0$ when

$$k^2 v_A^2 < 3\Omega^2. \quad (6.83)$$

In that case ω is plus or minus an imaginary number. Since the solution is proportional to $e^{-i\omega t}$ that means that the solution behaves like

$$e^{-i\omega t} = e^{\pm |\text{Im}(\omega)|t}. \quad (6.84)$$

The solution for which the upper sign applies grows fastest. We thus say that the solution is *unstable*. Eventually the magnitude of the velocity and magnetic field perturbations will be so large that the linearised equations are no longer valid.

Note that we were able to draw conclusions about stability and instability even though we did not actually solve the dispersion relation. This happens quite often, especially with higher order polynomials. But in the present case an explicit solution can easily be written down. It is

$$\omega_{1,2}^2 = v_A^2 k^2 + \frac{1}{2}\Omega^2 \pm \sqrt{4v_A^2 k^2 \Omega^2 + \frac{1}{4}\Omega^4} \quad (6.85)$$

We see that the presence of shear has a destabilising effect on the slow magnetosonic waves. This instability is also known as Balbus-Hawley instability and it has great relevance for causing turbulence in accretion discs.

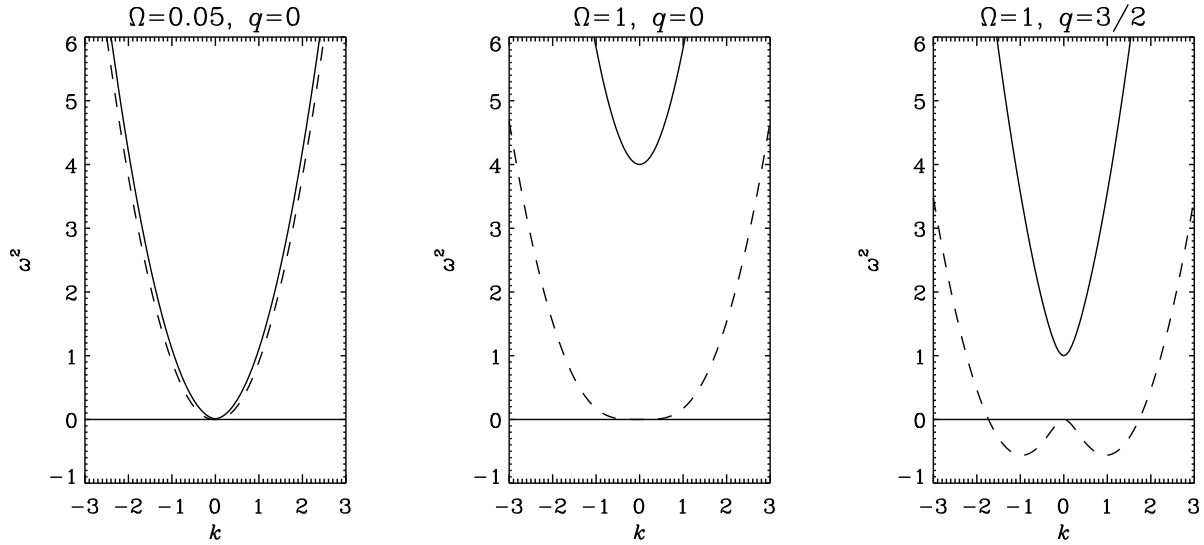


Figure 6.1: Dispersion relation for slow magnetosonic and Alfvén waves. Solid lines denote Alfvén waves. For them ω^2 is always positive. Dashed lines refer to slow magnetosonic waves. For not too large values of k they can become unstable ($\omega^2 < 0$, so ω becomes imaginary!).

6.3.3 Eigenfunction for the Balbus-Hawley instability

A simple way to get the eigenfunction of the Balbus-Hawley instability, or in fact of any linear eigenvalue problem, is to use the original equations starting with the most simple one. The simplest one is eq. (6.72), because it involves just 2 terms. We may choose the coefficient in front of $\hat{v}_x$ to one, so

$$\hat{v}_x = 1 \quad (6.86)$$

Using eq. (6.72) it follows that

$$\hat{B}_x = -\frac{ik}{i\omega} B_0 \quad (6.87)$$

Next we use eq. (6.70) to find $\hat{v}_y$

$$\hat{v}_y = -\frac{i\omega + \frac{B_0}{\rho_0\mu_0}ik \left(-\frac{ik}{i\omega}B_0\right)}{2\Omega} \quad (6.88)$$

or

$$\hat{v}_y = \frac{\omega^2 - v_A^2 k^2}{2\Omega i\omega} \quad (6.89)$$

Finally, we need to calculate $\hat{B}_y$, but we have two equations still unused, (6.71) and (6.73). We use first eq. (6.71) to obtain

$$\hat{B}_y = \frac{-i\omega\hat{v}_y + \frac{1}{2}\Omega\hat{v}_x}{ikv_A^2} B_0 \quad (6.90)$$

or, using the above results,

$$\hat{B}_y = \frac{-\left(\frac{\omega^2 - v_A^2 k^2}{2\Omega}\right) + \frac{1}{2}\Omega}{ikv_A^2} B_0 \quad (6.91)$$

so

$$\hat{B}_y = ikB_0 \frac{\omega^2 - v_A^2 k^2 - \Omega^2}{2\Omega k^2 v_A^2} \quad (6.92)$$

Finally, we can use the last equation, (6.73) to check everything, so we plug in the results obtained so far. We have

$$-i\omega \hat{B}_y = B_0 ik \hat{v}_y - \frac{3}{2}\Omega \hat{B}_x. \quad (6.93)$$

so

$$-i\omega(ikB_0) \frac{\omega^2 - v_A^2 k^2 - \Omega^2}{2\Omega k^2 v_A^2} = B_0 ik \frac{\omega^2 - v_A^2 k^2}{2\Omega i\omega} - \frac{3}{2}\Omega \left(-\frac{ik}{i\omega} B_0\right). \quad (6.94)$$

We cancel ikB_0 on both sides, and multiply by $i\omega$, so

$$\omega^2 \frac{\omega^2 - v_A^2 k^2 - \Omega^2}{2\Omega k^2 v_A^2} = \frac{\omega^2 - v_A^2 k^2}{2\Omega} + \frac{3}{2}\Omega. \quad (6.95)$$

Multiplying by $2\Omega k^2 v_A^2$ yields

$$\omega^2 (\omega^2 - v_A^2 k^2 - \Omega^2) = (\omega^2 - v_A^2 k^2) k^2 v_A^2 + 3\Omega^2 k^2 v_A^2. \quad (6.96)$$

which gives the old biquadratic equation,

$$\omega^4 - \omega^2(2v_A^2 k^2 - \Omega^2) + (k^2 v_A^2 - 3\Omega^2) k^2 v_A^2 = 0, \quad (6.97)$$

so everything seems to have gone alright. Anyway, the eigenfunction is therefore

$$\begin{pmatrix} \hat{v}_x \\ \hat{v}_y \\ \hat{B}_x \\ \hat{B}_y \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{\omega^2 - v_A^2 k^2}{2\Omega i\omega} \\ -\frac{ik}{i\omega} B_0 \\ ikB_0 \frac{\omega^2 - v_A^2 k^2 - \Omega^2}{2\Omega k^2 v_A^2} \end{pmatrix} \quad (6.98)$$

6.4 Nonaxisymmetric Balbus-Hawley instability

Due to shear, one expects a strong toroidal magnetic field in accretion discs. In that case ... $\mathbf{B}_0 = (0, B_0, 0)$, so

$$\mathbf{j} \times \mathbf{B}_0 = \mathbf{B}_0 \cdot \nabla \mathbf{b} - \nabla (\mathbf{B}_0 \cdot \mathbf{b}) = B_0 (\partial_y \mathbf{b} - \nabla b_y) = B_0 \begin{pmatrix} \partial_y & -\partial_x & 0 \\ 0 & 0 & 0 \\ 0 & -\partial_z & \partial_y \end{pmatrix} \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix} \quad (6.99)$$

$$\nabla \times (\mathbf{u} \times \mathbf{B}_0) = \mathbf{B}_0 \cdot \nabla \mathbf{u} - \mathbf{B}_0 \nabla \cdot \mathbf{u} = B_0 \begin{pmatrix} \partial_y & 0 & 0 \\ -\partial_x & 0 & -\partial_z \\ 0 & \partial_y & 0 \end{pmatrix} \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} \quad (6.100)$$

$$\nabla \times (\mathbf{U}_0 \times \mathbf{b}) = -\mathbf{U}_0 \cdot \nabla \mathbf{b} + \mathbf{b} \cdot \nabla \mathbf{U}_0 = -Sx\partial_y \mathbf{b} + S \begin{pmatrix} 0 \\ b_x \\ 0 \end{pmatrix} \quad (6.101)$$

In the general nonaxisymmetric case we can solve the problem by letting k_x be time-dependent, ie $k_x = k_x(t)$, with

$$k_x(t) = k_x(0) + q\Omega t. \quad (6.102)$$

In that case the solutions are of the form $\mathbf{q} = \hat{\mathbf{q}}(t) \exp[i\mathbf{k}(t) \cdot \mathbf{x}]$, and we are left with a system of *ordinary* differential equations,

$$\frac{d\hat{\mathbf{q}}}{dt} = -i\hat{\mathbf{L}}\hat{\mathbf{q}} \quad (6.103)$$

with the matrix

$$\hat{\mathbf{A}} \equiv -i\hat{\mathbf{L}}(t) = \begin{pmatrix} 0 & 2\Omega & 0 & ik_y & -ik_x(t) & 0 \\ -(2-q)\Omega & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -ik_z & ik_y \\ ik_y & 0 & 0 & 0 & 0 & 0 \\ -ik_x(t) & 0 & -ik_z & -q\Omega & 0 & 0 \\ 0 & 0 & ik_y & 0 & 0 & 0 \end{pmatrix} \quad (6.104)$$

being explicitly time-dependent. Note that in the absence of shear, $q = 0$, the matrix $\mathbf{A}$ is *antihermitian*, ie $A_{ij}^* = -A_{ji}$. This means that

$$\sum_{i=1}^6 q_i^* A_{ij} q_j \quad \text{is purely imaginary} \quad (6.105)$$

If either $q = 0$ or $k_x = 0$, we have the dispersion relation $-\omega^2 D(\omega, \mathbf{k}) = 0$ with

$$D = \omega^4 - \omega^2(\mathbf{k}^2 + k_y^2 + \kappa^2) + k_y^2 \mathbf{k}^2 + \kappa^2(k_y^2 + k_z^2). \quad (6.106)$$

In the general case, however, if one ignores the time dependence of $k_x(t)$, one has

$$i\omega D + (\omega^2 - k_y^2)q\Omega k_x k_y = 0. \quad (6.107)$$

6.5 Magnetosonic waves

We now consider the linearised MHD equations in vector notation,

$$-i\omega \hat{\mathbf{v}} = -i\mathbf{k}(c_s^2/\rho_0)\hat{\rho} - i\mathbf{k} \frac{\mathbf{B}_0 \cdot \hat{\mathbf{B}}}{\mu_0 \rho_0} + \frac{i\mathbf{k} \cdot \mathbf{B}_0}{\mu_0 \rho_0} \hat{\mathbf{B}}, \quad (6.108)$$

$$-i\omega \hat{\rho} = -i\mathbf{k} \cdot \hat{\mathbf{v}}, \quad (6.109)$$

$$-i\omega \hat{\mathbf{B}} = (i\mathbf{k} \cdot \mathbf{B}_0)\hat{\mathbf{v}} - (i\mathbf{k} \cdot \hat{\mathbf{v}})\mathbf{B}_0. \quad (6.110)$$

We multiply eq. (6.108) by $-\omega$ and substitute for $-\omega\hat{\rho}$ and $-\omega\hat{\mathbf{B}}$ using Eqs. (6.109) and (6.110), so

$$-\omega^2\hat{\mathbf{v}} = -i\mathbf{k}(c_s^2/\rho_0)(-i\mathbf{k}\cdot\hat{\mathbf{v}}) - \frac{i\mathbf{k}}{\mu_0\rho_0} \left[(i\mathbf{k}\cdot\mathbf{B}_0)(\hat{\mathbf{v}}\cdot\mathbf{B}_0) - (i\mathbf{k}\cdot\hat{\mathbf{v}})\mathbf{B}_0^2 \right] + \frac{i\mathbf{k}\cdot\mathbf{B}_0}{\mu_0\rho_0} [(i\mathbf{k}\cdot\mathbf{B}_0)\hat{\mathbf{v}} - (i\mathbf{k}\cdot\hat{\mathbf{v}})\mathbf{B}_0], \quad (6.111)$$

or

$$(k^2v_A^2 - \omega^2)\hat{\mathbf{v}} = +\mathbf{k} \left[-(\mathbf{k}\cdot\hat{\mathbf{v}})(c_s^2 + v_A^2) + (\mathbf{k}\cdot\mathbf{B}_0)(\hat{\mathbf{v}}\cdot\mathbf{B}_0)v_A^2 \right] + \hat{\mathbf{B}}_0(\mathbf{k}\cdot\mathbf{B}_0)(\mathbf{k}\cdot\hat{\mathbf{v}})v_A^2. \quad (6.112)$$

We now assume that the wave vector points in the x direction and that the magnetic field vector lies in the $x-y$ plane, i.e.

$$\mathbf{k} = \begin{pmatrix} k \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{B}_0 = \begin{pmatrix} \cos\psi \\ \sin\psi \\ 0 \end{pmatrix} B_0. \quad (6.113)$$

Therefore,

$$\hat{\mathbf{v}}\cdot\hat{\mathbf{B}}_0 = B_0(\hat{v}_x\cos\psi + \hat{v}_y\sin\psi), \quad \mathbf{k}\cdot\mathbf{B}_0 = kB_0\hat{v}_x\cos\psi, \quad \mathbf{k}\cdot\hat{\mathbf{v}} = k\hat{v}_x. \quad (6.114)$$

With this the z component of the right hand side vanishes, but because $\hat{v}_z$ does not generally vanish we have $k^2v_A^2 - \omega^2 = 0$, which gives already two possible solutions:

$$\omega = \pm kv_A. \quad (6.115)$$

For the remaining two components we have to satisfy the x and y components separately, i.e.

$$(k^2v_A^2 - \omega^2)\hat{v}_x = +k^2 \left[-\hat{v}_x(c_s^2 + v_A^2) + \cos\psi(\hat{v}_x\cos\psi + \hat{v}_y\sin\psi)v_A^2 \right] + \cos\psi(\hat{v}_x\cos\psi)\hat{v}_xk^2v_A^2. \quad (6.116)$$

$$(k^2v_A^2 - \omega^2)\hat{v}_y = +\sin\psi(\mathbf{k}\cdot\mathbf{B}_0)(\mathbf{k}\cdot\hat{\mathbf{v}})v_A^2. \quad (6.117)$$

This is a system of two algebraic equations. Written in matrix form we have, ..., which leads to the dispersion relation.... We note that there are altogether three wave types, slow and fast magnetosonic waves and Alfvén waves. The frequency depends on the values of k , the angle between the wave vector $\mathbf{k}$ and the magnetic field $\mathbf{B}_0$, and of course the values of the sound speed and the Alfvén speed. The dependence of the frequency as a function of the angle between $\mathbf{k}$ and $\mathbf{B}_0$ is shown in figure 6.2 for different values of c_s and v_A .

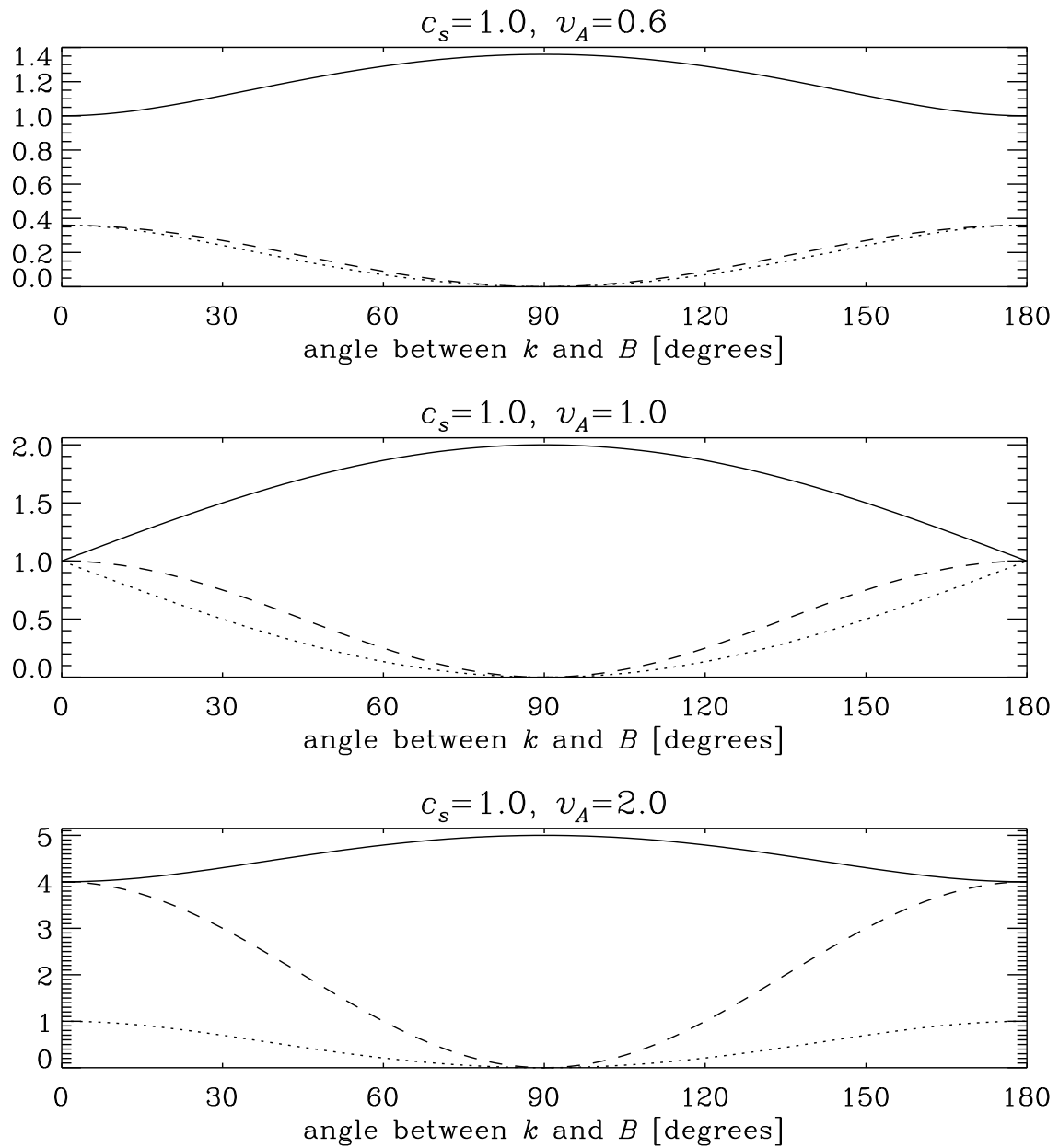


Figure 6.2: Dispersion relation for Alfvén waves. Solid lines denote the fast magnetosonic waves, dashed lines Alfvén waves, and dotted lines slow magnetosonic waves.

Chapter 7

Jeans instability

The Jeans instability is important for all processes involving the formation of massive localised bodies from a much more uniform background medium such as the interstellar medium, out of which stars have formed, the solar nebula, out of which planets have formed, and the intergalactic space, out of which galaxies have formed. All these processes operate on different scales and they often occur to some extent after each other, eg first the solar nebula and then the planets. Regarding the formation of galaxies and their (active) galactic nuclei it is still uncertain whether galaxies formed first, and then the stars within galaxies, or whether some massive stars may have formed prior to the first galaxies. A motivation for this latter possibility is that many active galactic nuclei (and Hubble space telescope observations strongly suggest that almost every single galaxy may have undergone an AGN stage at some early point in its evolution!) have high metallicities, and metals can only be generated during SN explosions.

7.1 Jeans instability without rotation

In this chapter we have to include selfgravity, ie the gravity resulting from a given density distribution. The gravity potential Φ is given by the Poisson equation,

$$\nabla^2 \Phi = 4\pi G \rho \quad (7.1)$$

where $G \approx 7 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}$ is Newton's gravitational constant. Although this equation is linear, there is actually a slight problem applying this to the case of a uniform background medium, because this would imply a potential that is diverging at infinity. This problem can be removed by considering a slowly varying background within which we will investigate the evolution of small amplitude waves.

The linearised continuity, Euler, and Poisson equations are then of the form

$$\frac{\partial \rho'}{\partial t} = -\rho_0 \nabla \cdot \mathbf{v}' \quad (7.2)$$

$$\frac{\partial \mathbf{v}'}{\partial t} = -\frac{c_s^2}{\rho_0} \nabla \rho' - \nabla \Phi' \quad (7.3)$$

$$\nabla \Phi' = 4\pi G \rho' \quad (7.4)$$

where primes indicate small perturbations about a (quasi-) uniform basic state, indicated by subscript 0. We rewrite those equations in terms of their Fourier transforms, $\hat{\rho}$, $\hat{\mathbf{v}}$, and $\hat{\Phi}$, assuming that all primed quantities are proportional to $\exp(i\mathbf{k} \cdot \mathbf{x} - i\omega t)$. Thus, we have

$$-i\omega \hat{\rho} = -i\mathbf{k} \hat{\mathbf{v}} \rho_0 \quad (7.5)$$

$$-i\omega \hat{\mathbf{v}} = -i\mathbf{k} \frac{c_s^2}{\rho_0} \hat{\rho} - i\mathbf{k} \hat{\Phi} \quad (7.6)$$

$$-\mathbf{k}^2 \hat{\Phi} = 4\pi G \hat{\rho} \quad (7.7)$$

We eliminate first $\hat{\mathbf{v}}$, so we multiply the first equation by $i\omega$,

$$\omega^2 \hat{\rho} = i\mathbf{k} \rho_0 (-i\omega \hat{\mathbf{v}}) \quad (7.8)$$

and get, using the second equation,

$$\omega^2 \hat{\rho} = i\mathbf{k} \rho_0 \left(-i\mathbf{k} \frac{c_s^2}{\rho_0} \hat{\rho} - i\mathbf{k} \hat{\Phi} \right), \quad (7.9)$$

or

$$\omega^2 \hat{\rho} = \mathbf{k}^2 c_s^2 \hat{\rho} + \mathbf{k}^2 \hat{\Phi} \rho_0. \quad (7.10)$$

Using now the third equation to eliminate $\hat{\Phi}$ we have

$$\omega^2 \hat{\rho} = \mathbf{k}^2 c_s^2 \hat{\rho} - 4\pi G \rho_0 \hat{\rho} \quad (7.11)$$

or

$$\left(\omega^2 - \mathbf{k}^2 c_s^2 + 4\pi G \rho_0 \right) \hat{\rho} = 0. \quad (7.12)$$

So, unless $\hat{\rho} = 0$, which would be the trivial case, we have to have

$$\omega^2 = \mathbf{k}^2 c_s^2 - 4\pi G \rho_0. \quad (7.13)$$

Contrast this with the dispersion relation for ordinary sound waves,

$$\omega^2 = \mathbf{k}^2 c_s^2, \quad (\text{ordinary sound waves}) \quad (7.14)$$

we notice that the significance of the second term is that it would destabilise those sound waves, provided

$$4\pi G \rho_0 > \mathbf{k}^2 c_s^2 \quad (\text{for instability}). \quad (7.15)$$

This could be the case either if the density is high or if the wave length of the perturbation is long. To calculate the critical wave length λ_{crit} we write

$$\lambda_{\text{crit}} = \frac{2\pi}{|\mathbf{k}|_{\text{crit}}} = \frac{2\pi c_s}{\sqrt{4\pi G \rho_0}} = \sqrt{\frac{\pi c_s^2}{G \rho_0}}. \quad (7.16)$$

As an example, for the typical density of the interstellar medium, $\rho = 2 \times 10^{-21} \text{ kg m}^{-3}$, and for $c_s = 10 \text{ km/s} = 10^4 \text{ ms}$ we have $\lambda_{\text{crit}} = 5 \times 10^{19} \text{ m} = 1.6 \text{ kpc}$. This is larger than the typical size of interstellar clouds. In reality clouds are compressed first by the turbulence, so they become denser, cool faster and so more realistic values are $\rho = 10^{-19} \text{ kg m}^{-3}$ and $c_s = 1 \text{ km/s}$ or less, so the corresponding size of the instability is then around 20 pc or less.

7.2 Jeans instability with rotation

In the presence of rotation one has to take the Coriolis force into account. The Euler equations takes then the following form

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{1}{\rho} \nabla p - 2\mathbf{\Omega} \times \mathbf{v} - \nabla \Phi, \quad (7.17)$$

where $\mathbf{\Omega} = (0, 0, \Omega)$ is the rotation vector and Ω is the (constant) rotation rate. Find the dispersion relation $\omega = \omega(\mathbf{k})$ assuming an isothermal equation of state, ie $p/\rho = c^2 = \text{const}$. Hint: write down the equations for all 3 components of $\mathbf{v}$ and the other 2 variables in matrix form (5×5 matrix!) and find where the determinant is zero.

The linearised, Fourier transformed equations in matrix form are

$$M_{ij} q_j = 0 \quad (7.18)$$

where

$$M_{ij} = \begin{pmatrix} -i\omega & ik_x \rho_0 & ik_y \rho_0 & ik_z \rho_0 & 0 \\ ik_x c^2 / \rho_0 & -i\omega & -2\Omega & 0 & ik_x \\ ik_y c^2 / \rho_0 & 2\Omega & -i\omega & 0 & ik_y \\ ik_z c^2 / \rho_0 & 0 & 0 & -i\omega & ik_z \\ 4\pi G & 0 & 0 & 0 & k^2 \end{pmatrix}, \quad q = \begin{pmatrix} \hat{\rho} \\ \hat{v}_x \\ \hat{v}_y \\ \hat{v}_z \\ \hat{\Phi} \end{pmatrix}. \quad (7.19)$$

To find nontrivial solutions the determinant of M_{ij} has to vanish. The determinant is given by

$$\mathbf{k}^{-2} \det M = \omega^4 - \omega^2(c^2 \mathbf{k}^2 + 4\Omega^2 - 4\pi G \rho_0) + 4\Omega^2 k_z^2 (c^2 \mathbf{k}^2 - 4\pi G \rho_0) / \mathbf{k}^2 = 0 \quad (7.20)$$

This is a quadratic equation in ω^2 . In the figure below we plot ω^2 in units of $\omega_0^2 \equiv 4\pi G \rho_0$ as a function of $\mathbf{k}^2$ for different values of Ω^2 . Note that the stability criterion is *not* changed, but instead a new (horizontal) branch appears which is independent of $\mathbf{k}$.

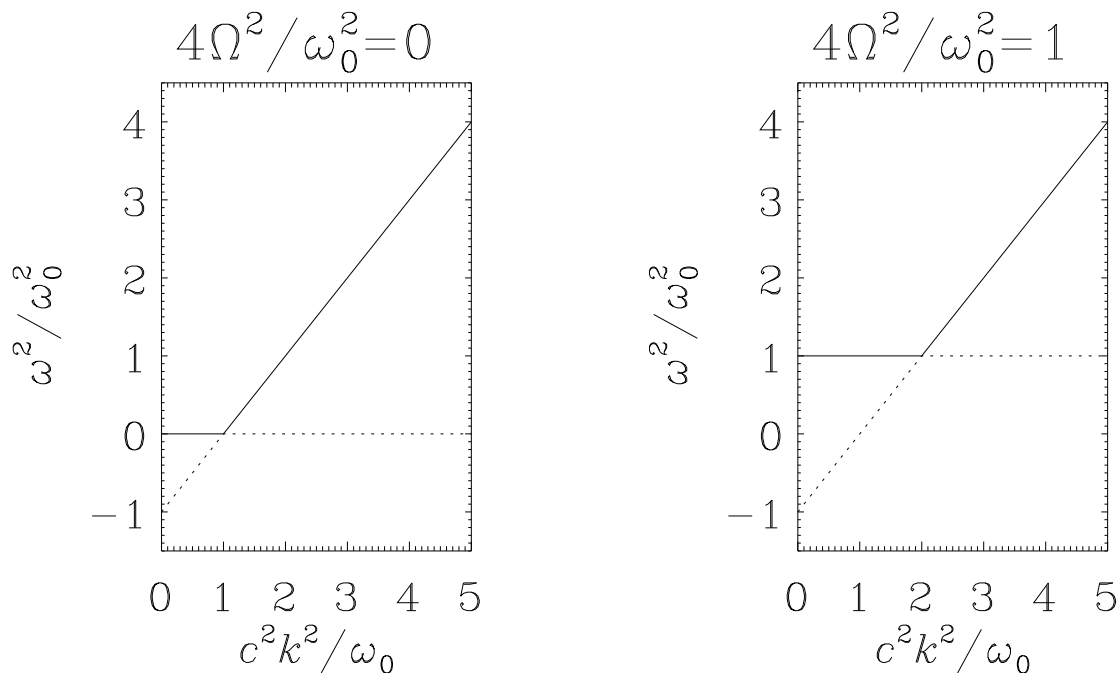


Figure 7.1: Dispersion relation. Solid and dotted lines refer to positive and negative signs in front of the square root of the solution.

7.3 Jeans instability in a disc

Consider the linearised continuity, Euler, and Poisson equations in the form

$$\frac{\partial \rho'}{\partial t} = -\rho_0 \nabla \cdot \mathbf{v}' \quad (7.21)$$

$$\frac{\partial \mathbf{v}'}{\partial t} = -\frac{c_s^2}{\rho_0} \nabla \rho' - \nabla \Phi' \quad (7.22)$$

$$\nabla \Phi' = 4\pi G \rho' \quad (7.23)$$

where primes indicate small perturbations about a uniform basic state, indicated by subscript 0. We rewrite those equations in terms of their Fourier transforms, $\hat{\rho}$, $\hat{\mathbf{v}}$, and $\hat{\Phi}$, assuming that all primed quantities are proportional to $\exp(i\mathbf{k} \cdot \mathbf{x} - i\omega t)$. We eliminate $\hat{\mathbf{v}}$ and $\hat{\Phi}$ to get the dispersion relation in the form $\omega^2(k^2)$.

Give a sketch of the dispersion relation. For which wave vectors is the medium unstable? Determine the maximum wavelength of a stable disturbance when $\rho_0 = 10^{-24} \text{ g/cm}^3$, $c_s = 1 \text{ km/s}$. (Note that $G \approx 7 \times 10^{-8} \text{ cm}^3 \text{ s}^{-2} \text{ g}^{-1}$ and $1 \text{ pc} \approx 3 \times 10^{18} \text{ cm}$.)

Consider the case of a thin disc with $\rho'(x, y, z) = \Sigma'(x, y)\delta(z)$, where $\Sigma'(x, y)$ is the perturbation

of the vertically integrated density. Show that Φ' has the form

$$\Phi' = \hat{\Phi} \exp(i\mathbf{k} \cdot \mathbf{x} - |\mathbf{k}|z - i\omega t), \quad (7.24)$$

where $\mathbf{k} = (k_x, k_y)$ and $\hat{\Phi} = \hat{\Phi}(k_x, k_y)$. Show that the Poisson equation yields

$$-|\mathbf{k}|\hat{\Phi} = 2\pi G\hat{\Sigma} \quad (7.25)$$

and that the final dispersion relation becomes

$$\omega^2 = c_s^2 k^2 - 2\pi G\Sigma_0 |\mathbf{k}|. \quad (7.26)$$

Chapter 8

Atmospheric waves

Before we come to the general case, we first consider one-dimensional sound waves without entropy perturbations. This means that we ignore Brunt-Väisälä oscillations. The main thing that we shall learn here is that it is important to bring the equations into a form such that it has constant coefficients.

8.1 Sound waves in a stratified atmosphere

In the presence of gravity there is density stratification which affects sound waves if their wave length becomes comparable to the density scale height. The one-dimensional continuity and momentum equations for an isothermal atmosphere with constant speed of sound, c_s , no entropy perturbations, and uniform gravity, g , can be written in the form

$$\rho \frac{\partial v_z}{\partial t} + \rho v_z \frac{\partial v_z}{\partial z} + c_s^2 \frac{\partial \rho}{\partial z} + \rho g = 0, \quad (8.1)$$

$$\frac{\partial \rho}{\partial t} + v_z \frac{\partial \rho}{\partial z} + \rho \frac{\partial v_z}{\partial z} = 0, \quad (8.2)$$

where ρ is density and v_z vertical velocity. With the exception of the ρg term in eq. (8.1), these equations are same as those used in §3.2 for ordinary sound waves. The hydrostatic equilibrium solution of these equations is $v_z = \overline{v_z} = 0$ and $\rho = \overline{\rho}(z) = \rho_0 e^{-z/H}$, where ρ_0 is a constant and $H = c_s^2/g$ is the vertical scale height. By writing $\rho = \overline{\rho} + \rho'$ and $v_z = v'_z$ one can linearise equations (8.1) and (8.2) with respect to $v_z = v'_z$ and ρ' about the solution $v_z = 0$ and $\rho = \overline{\rho}(z)$. This yields

$$\overline{\rho} \frac{\partial v'_z}{\partial t} + c_s^2 \frac{\partial \rho'}{\partial z} + \rho' g = 0, \quad (8.3)$$

$$\frac{\partial \rho'}{\partial t} + v'_z \frac{d\overline{\rho}}{dz} + \overline{\rho} \frac{\partial v'_z}{\partial z} = 0, \quad (8.4)$$

We may assume that $v_z = v'_z$ and ρ' take the form

$$v'_z(z, t) = \hat{v}_z e^{ikz - i\omega t + z/2H}. \quad (8.5)$$

$$\rho'(z, t) = \hat{\rho} e^{ikz - i\omega t - z/2H}, \quad (8.6)$$

The linearised equations can then be written as

$$\begin{pmatrix} -i\omega & [ik + (2H)^{-1}] \\ [ik - (2H)^{-1}]c_s^2 & -i\omega \end{pmatrix} \begin{pmatrix} \rho_0 \hat{v}_z \\ \hat{\rho} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (8.7)$$

The dispersion relation is then given by

$$\omega^2 = \omega_0^2 + k^2 c_s^2, \quad (8.8)$$

where we have introduced the acoustic cutoff frequency $\omega_0 = c_s/2H$. Figure 8.1 shows a plot of the dispersion relation.

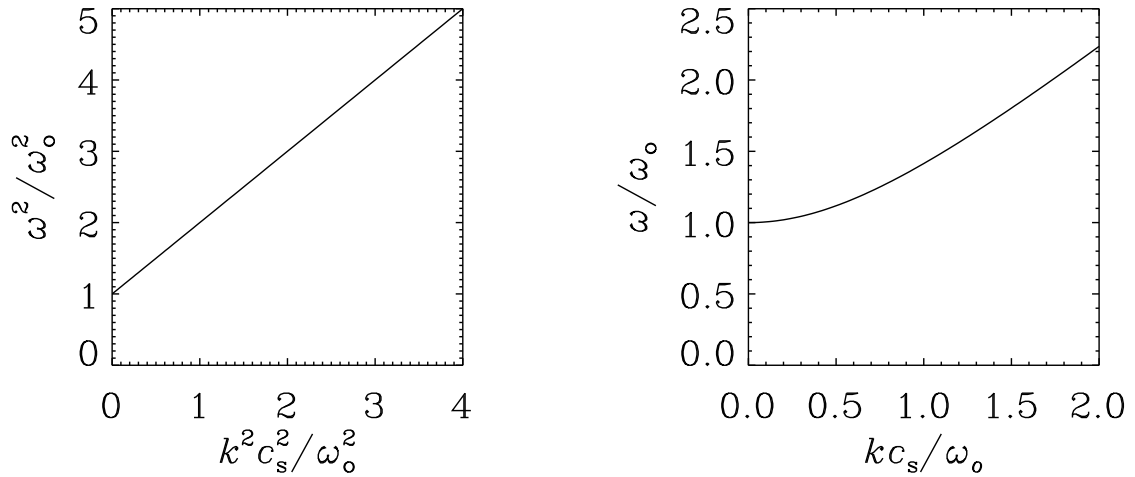


Figure 8.1: Dispersion relation for sound waves in a stratified atmosphere. On the left is shown $\omega^2(k^2)$ and on the right just $\omega(k)$.

As an example, we calculate the value of the period $T = 2\pi/\omega_0$ for the solar atmosphere, assuming $c_s = 6 \text{ km/s}$ and $g = 270 \text{ ms}^2$. First we have $H = c_s^2/g = (6 \times 10^3)/270 \text{ m} = 1.3 \times 10^5 \text{ m} = 130 \text{ km}$. Thus, $\omega_0 = c_s/2H = g/2c_s = 0.023 \text{ s}^{-1}$, so $T = 2\pi/0.023 \text{ s} = 280 \text{ s} \approx 5 \text{ min}$. This value is in fact close to the period of the 5 minutes oscillations of the sun.

8.2 What could have gone wrong

The linearized equations could have also been written in the form

$$\frac{\partial}{\partial t} \left(\frac{\rho'}{\bar{\rho}} \right) + v'_z \frac{d \ln \bar{\rho}}{dz} + \frac{\partial v'_z}{\partial z} = 0, \quad (8.9)$$

$$\frac{\partial v'_z}{\partial t} + c_s^2 \frac{\partial}{\partial z} \left(\frac{\rho'}{\bar{\rho}} \right) = 0, \quad (8.10)$$

Here we have made use of the fact that

$$\frac{\partial}{\partial z} \left(\frac{\rho'}{\bar{\rho}} \right) = -\frac{\rho'}{\bar{\rho}^2} \frac{d\bar{\rho}}{dz} + \frac{1}{\bar{\rho}} \frac{\partial \rho'}{\partial z} = +\frac{1}{H} \frac{\rho'}{\bar{\rho}} + \frac{1}{\bar{\rho}} \frac{\partial \rho'}{\partial z}, \quad (8.11)$$

Equations (8.9) and (8.10) have constant coefficients, so we can make the ansatz

$$\frac{\rho}{\bar{\rho}} = \hat{\vartheta} e^{ikz - i\omega t}, \quad (8.12)$$

$$v'_z(z, t) = \hat{v}_z e^{ikz - i\omega t}. \quad (8.13)$$

which leads to the matrix equation

$$\begin{pmatrix} -i\omega & ikc_s^2 \\ ik - H^{-1} & -i\omega \end{pmatrix} \begin{pmatrix} \hat{v}_z \\ \hat{\vartheta} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad (8.14)$$

For nontrivial solutions the determinant of the matrix has to vanish, which leads to the dispersion relation

$$\omega^2 = c_s^2 k^2 + ikc_s^2/H. \quad (8.15)$$

This result does not seem compatible with the previous one. In particular, instead of having stable waves, there now seems to be an instability: the frequency has an imaginary part with the growth rate $+kc_s^2/H$.

The reason for this apparent puzzle is that the eigenfunction is a wave whose amplitude increases with height. At any given point, however, the amplitude does not change. This is a classical case of what is called in Landau-Lifshitz *convective instability*¹ (this does have nothing to do with convection, however!).

In this case we can make ω real by allowing k to be complex. To see this we substitute

$$k = k' + ik'', \quad (8.16)$$

where k' and k'' refer to real and imaginary parts, respectively. Note that $k^2 = k'^2 + 2ik'k'' - k''^2$, so the dispersion relation becomes

$$\omega^2 = c_s^2(k'^2 - k''^2 - k''/H) + ik'c_s^2(2k'' + 1/H). \quad (8.17)$$

We now require that the imaginary part vanishes, ie $k'' = -1/(2H)$, and obtain

$$\omega^2 = c_s^2 \left(k'^2 + \frac{1}{4H^2} \right), \quad (8.18)$$

which is identical to eq. (8.8).

¹Physical Kinetics, Section VI

8.3 The general case

8.3.1 Basic equations

There are five basic equations describing the gas motions. These are the three momentum (or Euler) equations, the entropy equation (which was originally the energy equation) in the absence of heat sources or sinks, and the continuity equation, so

$$\rho \frac{D\mathbf{v}}{Dt} + \nabla p + \hat{\mathbf{z}}g\rho = 0, \quad (8.19)$$

$$\frac{Ds}{Dt} = 0, \quad (8.20)$$

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{v} = 0, \quad (8.21)$$

where we have assumed that gravity points only in the vertical direction and downwards, ie $\mathbf{g} = -g\hat{\mathbf{z}}$, where g is the gravitational acceleration (in m^2/s). We define the entropy as

$$\frac{s}{c_p} = \frac{1}{\gamma} \ln p - \ln \rho. \quad (8.22)$$

8.3.2 The equilibrium solution

The basic state is described by the simplest possible, and yet nontrivial, solution or solutions. For the basic state we denote all variables by a bar. In the present case the simplest solution is hydrostatic equilibrium, i.e. all quantities are independent of time, so $\partial/\partial t = 0$, and the velocity is also zero, $\bar{\mathbf{v}} = 0$. Thus, we have from eq. (8.19) simply

$$\frac{\partial \bar{p}}{\partial t} + \bar{\rho}g = 0. \quad (8.23)$$

We are interested in an isothermal atmosphere, because that case is far easier than other more general cases (such as a polytropic atmosphere). Therefore the sound speed is constant, so the square of the adiabatic sound speed is $c^2 = \gamma \bar{p}/\bar{\rho}$, or

$$\bar{\rho} = (\gamma/c^2) \bar{p}. \quad (8.24)$$

Thus, the hydrostatic equation becomes

$$\frac{\partial \bar{p}}{\partial t} = -\frac{\gamma g}{c^2} \bar{p}, \quad (8.25)$$

and the solution is

$$\bar{p} = p_0 \exp\left(-\frac{\gamma g z}{c^2}\right) \equiv p_0 \exp\left(-\frac{z}{H}\right), \quad (8.26)$$

where

$$H = \frac{c^2}{\gamma g} \quad (8.27)$$

is the scale height. Since the temperature is constant, the density is given by a similar law (with the same scale height), so

$$\bar{\rho} = \rho_0 \exp\left(-\frac{z}{H}\right). \quad (8.28)$$

The vertical derivative of the entropy for this hydrostatic case is

$$\frac{1}{c_p} \frac{d\bar{s}}{dz} = \frac{1}{\gamma} \frac{\partial \ln \bar{p}}{\partial z} - \frac{\partial \ln \bar{\rho}}{\partial z} = \left(1 - \frac{1}{\gamma}\right) \frac{1}{H}, \quad (8.29)$$

where γ is the ratio of specific heats ($\gamma = 5/3$ in many cases).

8.3.3 The continuity equation

The entropy is a useful quantity, because it satisfies a very simple equation. Keeping the pressure instead of the density has the advantage of having only one gradient term in the momentum equation. Thus, we express density now in terms of pressure and entropy. From eq. (8.22) we have

$$\frac{1}{c_p} \frac{Ds}{Dt} = \frac{1}{\gamma} \frac{D \ln p}{Dt} - \frac{D \ln \rho}{Dt}, \quad (8.30)$$

but since $Ds/Dt = 0$ in our case, we have

$$\frac{D \ln p}{Dt} = \gamma \frac{D \ln \rho}{Dt} \quad (8.31)$$

or, using eq. (8.21),

$$\frac{D \ln p}{Dt} = -\gamma \nabla \cdot \mathbf{v}, \quad (8.32)$$

which can also be written as

$$\frac{Dp}{Dt} + \gamma p \nabla \cdot \mathbf{v} = 0 \quad (8.33)$$

Now linearising this equation gives

$$\frac{\partial p'}{\partial t} + v'_z \frac{d\bar{p}}{dz} + \gamma \bar{p} \nabla \cdot \mathbf{v}' = 0 \quad (8.34)$$

For an isothermal atmosphere we have

$$\bar{p} = p_0 e^{-z/H}, \quad \text{and} \quad \frac{d\bar{p}}{dz} = -\frac{p_0}{H} e^{-z/H}, \quad (8.35)$$

so

$$\frac{\partial p'}{\partial t} - v'_z \frac{p_0}{H} e^{-z/H} + \gamma p_0 e^{-z/H} \nabla \cdot \mathbf{v}' = 0 \quad (8.36)$$

The problem with this equation is that it has non-constant coefficient. However, if we make the substitution [remember that $\bar{p} = \bar{p}(z)$]

$$\frac{p'}{\gamma\bar{p}} = \tilde{p}' \exp\left(\frac{z}{2H}\right), \quad \frac{s'}{c_p} = \tilde{s}' \exp\left(\frac{z}{2H}\right), \quad \frac{\mathbf{v}'}{c_s} = \tilde{\mathbf{v}}' \exp\left(\frac{z}{2H}\right). \quad (8.37)$$

we are able to remove all z -dependent coefficients, so we have

$$\frac{\partial \tilde{p}'}{\partial t} - \left(1 - \frac{\gamma}{2}\right) \frac{1}{\gamma H} \tilde{v}'_z + \nabla \cdot \tilde{\mathbf{v}}' = 0, \quad (8.38)$$

which is now an equation with *constant coefficients*. Note that this scaling implies that velocity and entropy perturbations increase with height, but the pressure perturbations decrease with height because of the $\bar{p}(z)$ dependence. The last equation can also be written in the form

$$\frac{\partial \tilde{p}'}{\partial t} - \left(1 - \frac{\gamma}{2}\right) \frac{g}{c_s^2} \tilde{v}'_z + \nabla \cdot \tilde{\mathbf{v}}' = 0, \quad (8.39)$$

8.3.4 The momentum equation

We start from the usual momentum equation with constant vertical acceleration due to gravity, so

$$\rho \frac{D\mathbf{v}}{Dt} + \nabla p + \hat{\mathbf{z}}\rho g = 0 \quad (8.40)$$

Linearising this gives

$$\bar{\rho} \frac{\partial \mathbf{v}'}{\partial t} + \nabla p' + \hat{\mathbf{z}}\rho' g = 0, \quad (8.41)$$

or

$$\frac{\partial \mathbf{v}'}{\partial t} + \frac{1}{\bar{\rho}} \nabla p' + \hat{\mathbf{z}}g \frac{\rho'}{\bar{\rho}} = 0, \quad (8.42)$$

We use as dependent thermodynamic variables p and s , so we need to express ρ in terms of p and s . We use the definition of s in the form

$$\frac{s}{c_p} = \frac{1}{\gamma} \ln p - \ln \rho \quad (8.43)$$

For the deviation from the basic state we have then

$$\frac{s'}{c_p} = \frac{p'}{\gamma\bar{p}} - \frac{\rho'}{\bar{\rho}}, \quad (8.44)$$

so in the momentum equation above we eliminate $\rho'/\bar{\rho}$ and note that

$$\frac{1}{\bar{\rho}} \nabla p' = \frac{1}{\bar{\rho}} \nabla \left(\tilde{p}' \gamma \bar{p} e^{z/2H} \right) = \frac{\gamma \bar{p}}{\bar{\rho}} e^{z/2H} \left(\nabla \tilde{p}' - \frac{1}{H} + \frac{1}{2H} \right) = c_s^2 e^{z/2H} \left(\nabla \tilde{p}' - \frac{1}{2H} \right), \quad (8.45)$$

and obtain

$$\frac{\partial \mathbf{v}'}{\partial t} + c_s^2 \nabla \tilde{p}' - \frac{c_s^2}{2H} \tilde{p}' + \hat{\mathbf{z}}g \tilde{p}' - \hat{\mathbf{z}}g \tilde{s}' = 0. \quad (8.46)$$

which has now constant coefficients. Remembering that $c_s^2 = \gamma H g$ we can combine $g - c_s^2/2H$ to $(1 - \gamma/2)g$ and have

$$\frac{\partial \tilde{\mathbf{v}}'}{\partial t} + c_s^2 \nabla \tilde{p}' + \hat{\mathbf{z}}g \left(1 - \frac{\gamma}{2}\right) \tilde{p}' - \hat{\mathbf{z}}g \tilde{s}' = 0. \quad (8.47)$$

8.3.5 The entropy equation

Finally, the linearised entropy equation, $Ds/Dt = 0$, becomes

$$\frac{\partial s'}{\partial t} + v'_z \frac{d\bar{s}}{dz} = 0, \quad (8.48)$$

where the expression for $d\bar{s}/dz$ was given in eq. (8.29), so

$$\frac{\partial s'}{\partial t} + \left(1 - \frac{1}{\gamma}\right) \frac{v'_z}{H} = 0. \quad (8.49)$$

This equation has constant coefficients already, and this property does not change if we go over to the transformed variables with tildes (both $\tilde{s}'$ and $\tilde{v}'_z$ have the same scaling factor), so

$$\frac{\partial \tilde{s}'}{\partial t} + (\gamma - 1) \frac{g}{c_s} \tilde{v}'_z = 0. \quad (8.50)$$

8.3.6 The nondimensional set of equations

The equations look a bit messy. We are already using the nondimensional dependent variables $\tilde{\mathbf{v}}'$, $\tilde{s}'$, $\tilde{p}'$. Here we also introduce nondimensional measures for time and space coordinates. (with a tilde) as

$$\frac{\partial}{\partial t} = \frac{g}{c_s} \frac{\partial}{\partial \tilde{t}}, \quad \nabla = \frac{g}{c_s^2} \hat{\nabla}. \quad (8.51)$$

With these preparations we obtain now the following set of equations:

$$\frac{\partial \tilde{\mathbf{v}}'}{\partial \tilde{t}} + \tilde{\nabla} \tilde{p}' + \hat{\mathbf{z}} \left(1 - \frac{\gamma}{2}\right) \tilde{p}' - \hat{\mathbf{z}} \tilde{s}' = 0. \quad (8.52)$$

$$\frac{\partial \tilde{s}'}{\partial \tilde{t}} + (\gamma - 1) \tilde{v}'_z = 0. \quad (8.53)$$

$$\frac{\partial \tilde{p}'}{\partial \tilde{t}} - \left(1 - \frac{\gamma}{2}\right) \tilde{v}'_z + \tilde{\nabla} \cdot \tilde{\mathbf{v}}' = 0, \quad (8.54)$$

We can even go one step further by dividing all three equations by $(1 - \gamma/2)$ and *revising* our definition of nondimensional time and space coordinates,

$$\frac{\partial}{\partial t} = \left(1 - \frac{\gamma}{2}\right) \frac{g}{c_s} \frac{\partial}{\partial \tilde{t}}, \quad \nabla = \left(1 - \frac{\gamma}{2}\right) \frac{g}{c_s^2} \hat{\nabla}. \quad (8.55)$$

and *redefining* the nondimensional entropy as

$$\frac{s'}{c_p} = \sqrt{\gamma - 1} \tilde{s}' \exp\left(\frac{z}{2H}\right), \quad (8.56)$$

This yields the equations

$$\frac{\partial \tilde{\mathbf{v}}'}{\partial \tilde{t}} + \tilde{\nabla} \tilde{p}' + \hat{\mathbf{z}} \tilde{p}' - \hat{\mathbf{z}} \omega_{\text{BV}} \tilde{s}' = 0. \quad (8.57)$$

$$\frac{\partial \tilde{s}'}{\partial \tilde{t}} + \omega_{\text{BV}} \tilde{v}'_z = 0, \quad (8.58)$$

$$\frac{\partial \tilde{p}'}{\partial \tilde{t}} - \tilde{v}'_z + \tilde{\nabla} \cdot \tilde{\mathbf{v}}' = 0, \quad (8.59)$$

where $\omega_{\text{BV}} = \sqrt{\gamma - 1}/(1 - \gamma/2)$. The advantage of this somewhat unnecessary procedure is of purely esthetic nature and becomes clear when writing the final equations in matrix form, $\mathbf{i}\dot{\mathbf{q}} = \mathbf{L}\mathbf{q}$, with

$$\mathbf{L} = \begin{pmatrix} 0 & 0 & 0 & 0 & -i\partial_x \\ 0 & 0 & 0 & 0 & -i\partial_y \\ 0 & 0 & 0 & i\omega_{\text{BV}} & -i - i\partial_z \\ 0 & 0 & -i\omega_{\text{BV}} & 0 & 0 \\ -i\partial_x & -i\partial_y & i - i\partial_z & 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{q} = \begin{pmatrix} \tilde{v}'_x \\ \tilde{v}'_y \\ \tilde{v}'_z \\ \tilde{s}' \\ \tilde{p}' \end{pmatrix}. \quad (8.60)$$

The operator $\mathbf{L}$ is hermitian and has therefore real eigenvalues. This allows us to say, without having solved the actual eigenvalue problem, that the system is stable!

Next we calculate the dispersion relation by making the ansatz $\mathbf{q} = \hat{\mathbf{q}} \exp(i\mathbf{k} \cdot \mathbf{x} - i\omega t)$. We arrive then at an algebraic system of equations,

$$\omega \hat{\mathbf{q}} = \hat{\mathbf{L}} \hat{\mathbf{q}}, \quad (8.61)$$

where ∂_t is replaced by $-i\omega$ and ∂_x is replaced by ik_x , etc. As usual, non-trivial solutions require that the determinant of the matrix $\hat{\mathbf{A}} \equiv \hat{\mathbf{L}} + \omega \mathbf{I}$ vanishes. Here, $\mathbf{I}$ is the unit matrix and

$$\hat{\mathbf{A}} = \begin{pmatrix} -\omega & 0 & 0 & 0 & k_x \\ 0 & -\omega & 0 & 0 & k_y \\ 0 & 0 & -\omega & i\omega_{\text{BV}} & -i + k_z \\ 0 & 0 & -i\omega_{\text{BV}} & -\omega & 0 \\ k_x & k_y & i + k_z & 0 & -\omega \end{pmatrix} \quad (8.62)$$

To make sure we get it right, it helps to simplify matters. The system is symmetric with respect to k_x and k_y , so we consider first the reduced two-dimensional problem, governed by the matrix

$$\hat{\mathbf{A}} = \begin{pmatrix} -\omega & 0 & 0 & k_x \\ 0 & -\omega & i\omega_{\text{BV}} & -i + k_z \\ 0 & -i\omega_{\text{BV}} & -\omega & 0 \\ k_x & i + k_z & 0 & -\omega \end{pmatrix} \quad (8.63)$$

We now calculate the determinant,

$$\det \hat{\mathbf{A}} = -\omega \det \begin{pmatrix} -\omega & i\omega_{\text{BV}} & -i + k_z \\ -i\omega_{\text{BV}} & -\omega & 0 \\ i + k_z & 0 & -\omega \end{pmatrix} - k_x \det \begin{pmatrix} 0 & -\omega & i\omega_{\text{BV}} \\ 0 & -i\omega_{\text{BV}} & -\omega \\ k_x & i + k_z & 0 \end{pmatrix} = 0, \quad (8.64)$$

so

$$\omega^2(\omega^2 - k_z^2 - 1) - k_x^2(\omega^2 - \omega_{\text{BV}}^2) = 0 \quad (8.65)$$

or

$$\omega^4 - \omega^2(k^2 + \omega_{\text{BV}}^2 + 1) + k_{\perp}^2 \omega_{\text{BV}}^2 = 0, \quad (8.66)$$

where $k^2 = k_x^2 + k_z^2$ and $k_{\perp}^2 = k_y^2$. If $k_y \neq 0$, this dispersion relation is still valid if we set

$$k^2 = k_x^2 + k_y^2 + k_z^2, \quad k_{\perp}^2 = k_x^2 + k_y^2. \quad (8.67)$$

To convert (8.68) back into dimensional units, we just have to make sure that all dimensions match. In front of the k^2 factor, for example, we have to have a c_s^2 to match the dimensions of ω^2 . Likewise, unity has to be made into something with the same dimension as ω^2 . According to eq. (8.55), unity becomes $(1 - \frac{1}{2}\gamma)^2 g^2 / c_s^2$ in order to match the dimensions of ω^2 . To turn the dimensionless ω_{BV}^2 term into the same dimension we have to multiply $(\gamma - 1)/(1 - \gamma/2)^2$ with $(1 - \frac{1}{2}\gamma)^2 g^2 / c_s^2$ and get just $(\gamma - 1)g^2 / c_s^2$. Thus the dispersion relation in dimensional units reads

$$\omega^4 - \omega^2 \left[c_s^2 k^2 + (\gamma - 1) \frac{g^2}{c_s^2} + \left(1 - \frac{\gamma}{2}\right)^2 \frac{g^2}{c_s^2} \right] + k_{\perp}^2 g^2 (\gamma - 1) = 0, \quad (8.68)$$

which can be simplified to give

$$\omega^4 - \omega^2 \left(c_s^2 k^2 + \frac{\gamma^2 g^2}{4 c_s^2} \right) + k_{\perp}^2 g^2 (\gamma - 1) = 0. \quad (8.69)$$

By writing $k_{\perp} = k \sin \theta$, where θ is the polar angle of the wave vector, we can express the dispersion relation in the form

$$k^2 = \frac{\omega^2}{c_s^2} \frac{\omega^2 - N_{\text{ac}}^2}{\omega^2 - N_{\text{BV}}^2 \sin^2 \theta}, \quad \text{where} \quad N_{\text{ac}} = \frac{\gamma g}{2 c_s} \quad \text{and} \quad N_{\text{BV}} = \sqrt{\gamma - 1} \frac{g}{c_s}, \quad (8.70)$$

are the acoustic and Brunt-Väisälä frequencies in dimensional form.

8.3.7 The anelastic approximation: filtering out sound waves

In the anelastic approximation one *replaces* the continuity equation by $\nabla \cdot (\rho \mathbf{v}) = 0$. This yields the matrix

$$\hat{\mathbf{A}} = \begin{pmatrix} -\omega & 0 & 0 & k_x \\ 0 & -\omega & i\omega_{\text{BV}} & -i + k_z \\ 0 & -i\omega_{\text{BV}} & -\omega & 0 \\ k_x & i + k_z & 0 & 0 \end{pmatrix} \quad (8.71)$$

which yields the dispersion relation

$$-\omega^2(k^2 + 1) + k_{\perp}^2 \omega_{\text{BV}}^2 = 0. \quad (8.72)$$

In dimensional units the dispersion relation reads

$$-\omega^2 \left[c_s^2 k^2 + \left(1 - \frac{\gamma}{2}\right)^2 \frac{g^2}{c_s^2} \right] + k_{\perp}^2 g^2 (\gamma - 1) = 0. \quad (8.73)$$

8.3.8 The isothermal case: filtering out g-modes

In the isothermal case, $\gamma = 1$, we have $\omega_{\text{BV}} = 0$, and so the dispersion relation becomes $\omega^2 = k^2 + 1$, which is just the same as eq. (8.8).

8.4 Rotation

In the presence of rotation we have to add the Coriolis force, so the matrix is then

$$\hat{\mathbf{A}} = \begin{pmatrix} -\omega & 2i\Omega & 0 & 0 & k_x \\ -2i\Omega & -\omega & 0 & 0 & k_y \\ 0 & 0 & -\omega & i\omega_{\text{BV}} & -i + k_z \\ 0 & 0 & -i\omega_{\text{BV}} & -\omega & 0 \\ k_x & k_y & i + k_z & 0 & -\omega \end{pmatrix} \quad (8.74)$$

which yields the dispersion relation (see figure 8.2)

$$\omega^4 - \omega^2(k^2 + \omega_{\text{BV}}^2 + 4\Omega^2 + 1) + k_{\perp}^2 \omega_{\text{BV}}^2 + 4\Omega^2(k_z^2 + \omega_{\text{BV}}^2 + 1) = 0, \quad (8.75)$$

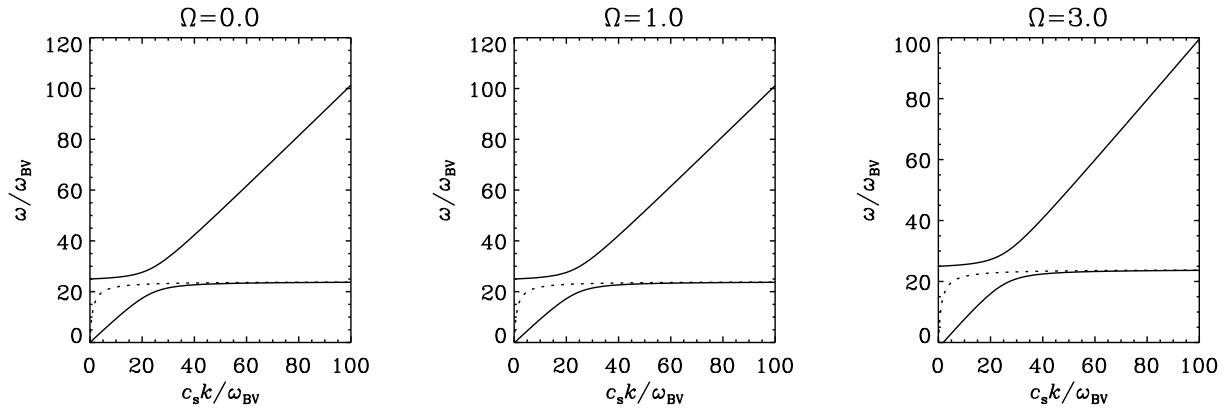


Figure 8.2: Dispersion relation for a stratified atmosphere under the presence of rotation. The upper and lower branches correspond to sound and gravity waves, respectively. $k_z = 0$, $\omega_{\text{BV}} = 4.9$ (corresponding to $\gamma = 5/3$). The dotted line gives the dispersion relation in the anelastic approximation.

In the anelastic approximation the dispersion relation simplifies to

$$-\omega^2(k^2 + 1) + k_{\perp}^2 \omega_{\text{BV}}^2 + 4\Omega^2(k_z^2 + 1) = 0, \quad (8.76)$$

whilst the isothermal case leads to

$$\omega^4 - \omega^2(k^2 + 4\Omega^2 + 1) + 4\Omega^2(k_z^2 + 1) = 0, \quad (8.77)$$

8.5 Rotation and shear

In the presence of shear and rotation we have in general a problem with variable (x -dependent) coefficients. Only in the axisymmetric case, corresponding to $k_y = 0$, do we still have constant coefficients. In that case the matrix reads

$$\hat{\mathbf{A}} = \begin{pmatrix} -\omega & 2i\Omega & 0 & 0 & k_x \\ -(2-q)i\Omega & -\omega & 0 & 0 & 0 \\ 0 & 0 & -\omega & i\omega_{\text{BV}} & -i + k_z \\ 0 & 0 & -i\omega_{\text{BV}} & -\omega & 0 \\ k_x & 0 & i + k_z & 0 & -\omega \end{pmatrix} \quad (8.78)$$

where $q = 3/2$ for keplerian shear. This yields the dispersion relation

$$\omega^4 - \omega^2(k^2 + \omega_{\text{BV}}^2 + \kappa^2 + 1) + k_x^2 \omega_{\text{BV}}^2 + \kappa^2(k_z^2 + \omega_{\text{BV}}^2 + 1) = 0, \quad (8.79)$$

where we have used the epicyclic frequency, κ , which is given by $\kappa^2 = 2(2-q)\Omega^2$. This dispersion relation agrees with eq. (8.77) if we replace 2Ω by κ . The flow becomes unstable when $q > 0$, ie $\kappa^2 < 0$. This corresponds to the Rayleigh unstable case.

In the general nonaxisymmetric case we can solve the problem by letting k_x be time-dependent, ie $k_x = k_x(t)$, with

$$k_x(t) = k_x(0) + q\Omega t. \quad (8.80)$$

In that case the solutions are of the form $\mathbf{q} = \hat{\mathbf{q}}(t) \exp[i\mathbf{k}(t) \cdot \mathbf{x}]$, and we are left with a system of *ordinary* differential equations,

$$\frac{d\hat{\mathbf{q}}}{dt} = -i\hat{\mathbf{L}}\hat{\mathbf{q}} \quad (8.81)$$

with the matrix

$$\hat{\mathbf{A}} \equiv -i\hat{\mathbf{L}}(t) = \begin{pmatrix} 0 & 2\Omega & 0 & 0 & -ik_x(t) \\ -(2-q)\Omega & 0 & 0 & 0 & -ik_y \\ 0 & 0 & 0 & \omega_{\text{BV}} & -1 - ik_z \\ 0 & 0 & -\omega_{\text{BV}} & 0 & 0 \\ -ik_x(t) & -ik_y & 1 - ik_z & 0 & 0 \end{pmatrix} \quad (8.82)$$

being explicitly time-dependent. Note that in the absence of shear, $q = 0$, the matrix $\mathbf{A}$ is *antihermitian*, ie $A_{ij}^* = -A_{ji}$. This means that

$$\sum_{i=1}^5 q_i^* A_{ij} q_j \quad \text{is purely imaginary} \quad (8.83)$$

8.6 Ignoring time-dependence of k_x

If one ignores the time-dependence of k_x one obtains a *pseudo-dispersion relation*,

$$\omega^4 - \omega^2(k^2 + \omega_{\text{BV}}^2 + \kappa^2 + 1) + k_{\perp}^2 \omega_{\text{BV}}^2 + \kappa^2(k_z^2 + \omega_{\text{BV}}^2 + 1) + \frac{q\Omega}{i\omega} k_x k_y (\omega^2 - \omega_{\text{BV}}^2) = 0. \quad (8.84)$$

In the anelastic approximation, the pseudo-dispersion relation becomes

$$-\omega^2(k^2 + 1) + k_\perp^2 \omega_{\text{BV}}^2 + \kappa^2(k_z^2 + 1) + \frac{q\Omega}{i\omega} k_x k_y (\omega^2 - \omega_{\text{BV}}^2) = 0. \quad (8.85)$$

In the Boussinesq limit [ignoring in the matrix (8.88) the terms $\pm i$ in front of k_z] we have

$$-\omega^2 k^2 + k_\perp^2 \omega_{\text{BV}}^2 + \kappa^2 k_z^2 + \frac{q\Omega}{i\omega} k_x k_y (\omega^2 - \omega_{\text{BV}}^2) = 0. \quad (8.86)$$

or, after multiplying with $-\omega^2$,

$$\omega^4 k^2 - \omega^2 \left[k_\perp^2 \omega_{\text{BV}}^2 + \kappa^2 k_z^2 + i\omega q \Omega k_x k_y \right] + i\omega q \Omega k_x k_y \omega_{\text{BV}}^2 = 0. \quad (8.87)$$

8.7 Rotation, shear, and magnetic fields

Adding a *vertical* magnetic field yields the matrix equation

$$\hat{\mathbf{A}} = \begin{pmatrix} -\omega & 2i\Omega & 0 & 0 & k_x & -v_A k_z & 0 & v_A k_x \\ -(2-q)\Omega & -\omega & 0 & 0 & 0 & 0 & v_A k_z & 0 \\ 0 & 0 & -\omega & i\omega_{\text{BV}} & -i + k_z & 0 & 0 & 0 \\ 0 & 0 & -i\omega_{\text{BV}} & -\omega & 0 & 0 & 0 & 0 \\ k_x & 0 & i + k_z & 0 & -\omega & 0 & 0 & 0 \\ -v_A k_z & 0 & 0 & 0 & 0 & -\omega & 0 & 0 \\ 0 & -v_A k_z & 0 & 0 & 0 & -q\Omega & -\omega & 0 \\ v_A k_x & 0 & 0 & 0 & 0 & 0 & 0 & -\omega \end{pmatrix} \quad (8.88)$$

This yields the dispersion relation

$$\omega^4 - \omega^2(k^2 + \omega_{\text{BV}}^2 + \kappa^2 + 1) + k_x^2 \omega_{\text{BV}}^2 + \kappa^2(k_z^2 + \omega_{\text{BV}}^2 + 1) = 0, \quad (8.89)$$

where we have used the epicyclic frequency, κ , which is given by $\kappa^2 = 2(2-q)\Omega^2$. This dispersion relation agrees with eq. (8.77) if we replace 2Ω by κ . The flow becomes unstable when $q > 0$, ie $\kappa^2 < 0$. This corresponds to the Rayleigh unstable case.

In the general nonaxisymmetric case we can solve the problem by letting k_x be time-dependent, ie $k_x = k_x(t)$, with

$$k_x(t) = k_x(0) + q\Omega t. \quad (8.90)$$

In that case the solutions are of the form $\mathbf{q} = \hat{\mathbf{q}}(t) \exp[i\mathbf{k}(t) \cdot \mathbf{x}]$, and we are left with a system of *ordinary* differential equations,

$$\frac{d\hat{\mathbf{q}}}{dt} = -i\hat{\mathbf{L}}\hat{\mathbf{q}} \quad (8.91)$$

with the matrix

$$\hat{\mathbf{A}} \equiv -i\hat{\mathbf{L}}(t) = \begin{pmatrix} 0 & 2\Omega & 0 & 0 & -ik_x(t) \\ -(2-q)\Omega & 0 & 0 & 0 & -ik_y \\ 0 & 0 & 0 & \omega_{\text{BV}} & -1 - ik_z \\ 0 & 0 & -\omega_{\text{BV}} & 0 & 0 \\ -ik_x(t) & -ik_y & 1 - ik_z & 0 & 0 \end{pmatrix} \quad (8.92)$$

being explicitly time-dependent. Note that in the absence of shear, $q = 0$, the matrix $\mathbf{A}$ is *antihermitian*, ie $A_{ij}^* = -A_{ji}$. This means that

$$\sum_{i=1}^5 q_i^* A_{ij} q_j \quad \text{is purely imaginary} \quad (8.93)$$

Chapter 9

Appendix

.1 Taylor expansion of the Roche potential

We want to find the Taylor expansion of the Roche potential near the inner Lagrange point L_1 in the form

$$\Phi(x) \approx \Phi_0 + \frac{1}{2} \left(\frac{\partial^2 \Phi}{\partial x^2} \right)_{L_1} (x - x_{L_1})^2. \quad (1)$$

The Roche potential is

$$\Phi = -\frac{GM_1}{r_1} - \frac{GM_2}{r_2} - \frac{\omega^2 r_s^2}{2} \quad (2)$$

where

$$\omega^2 = \frac{G(M_1 + M_2)}{d^3} \quad (3)$$

and $d = R_1 + R_2$ is the distance between the two stars, and R_1 and R_2 are the distances from the centre of mass. From $R_1 M_1 = R_2 M_2$ we obtain

$$R_2 = \frac{d}{1 + \frac{M_2}{M_1}} \quad (4)$$

At a point (x, y) we have

$$r_1 = \sqrt{(x - d)^2 + y^2}, \quad r_2 = \sqrt{x^2 + y^2}, \quad r_s = \sqrt{(x - R_2)^2 + y^2} \quad (5)$$

Now Taylor expansion yields

$$\Phi(x) \approx \Phi_0 + \Phi_2 (x - x_{L_1})^2. \quad (6)$$

The location of L_1 is given by $\partial\Phi/\partial x = 0$ in the range $0 < x_{L_1} < d$. Put $y = 0$, so

$$\Phi(x) = -\frac{GM_1}{|x - d|} - \frac{GM_2}{|x|} - \frac{\omega^2 (x - R_2)^2}{2}. \quad (7)$$

Since $0 < x_{L_1} < d$ we may write

$$\Phi(x) = +\frac{GM_1}{x - d} - \frac{GM_2}{x} - \frac{\omega^2 (x - R_2)^2}{2}. \quad (8)$$

Differentiation yields

$$\frac{\partial \Phi}{\partial x} = -\frac{GM_1}{(x-d)^2} + \frac{GM_2}{x^2} - \omega^2(x - R_2). \quad (9)$$

Let's adopt units such that $G = M_1 = d = 1$ and assume $M_2 = 0.3$. With those numbers we find the value of x_{L_1} after 5 steps of iteration:

Table 1: Iteration results to find the value of x_{L_1} .

x	$\partial \Phi / \partial x$
0.3	1.90252
0.4	-0.422778
0.381818	-0.055325
0.379081	0.0010840
0.379133	0

So $x_{L_1} = 0.379$ and $\Phi_0 = \Phi(x_{L_1}) = -2.50$.

To find Φ_2 differentiate once more:

$$\frac{\partial^2 \Phi}{\partial x^2} = 2\frac{GM_1}{(x-d)^3} - 2\frac{GM_2}{x^3} - \omega^2. \quad (10)$$

At $x_{L_1} = 0.379$ we have $\Phi_2 = \frac{1}{2} \frac{\partial^2 \Phi}{\partial x^2} = -10.3$. In figure 2.6 we have plotted $\Phi(x)$ (solid line) together with the Taylor expansion (dotted line).

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